Problem A: (30 pts) Answer briefly the following questions. Each question has two points.

1. (Questions 1 to 5) Consider the daily log returns of Qualcomm stock from May 17, 2007 to April 30, 2013. Summary statistics of the log returns are given in the R output. Is the mean log return significantly different from zero? State the null and alternative hypotheses and draw the conclusion.

Answer: $H_0 : \mu = 0$ versus $H_a : \mu \neq 0$. Since the 95% interval for the mean contains zero, one cannot reject the null hypothesis. Therefore, the mean log return is not significantly different from zero.

2. Does the log returns have heavy tails? Perform a test and draw the conclusion.

Answer: Yes. The $t$-ratio of the excess kurtosis is $t = 7.1796/\sqrt{24/1499} = 56.74$, which is highly significant with $p$-value close to zero.

3. What is the standard error (SE) of the mean log returns? What is the $t$-ratio ($t = \frac{\bar{r}}{\text{std}(\bar{r})}$) of the sample mean $\bar{r}$ of the log return $r_t$?

Answer: SE = 0.021824/\sqrt{1499} = 0.000564. The $t$-ratio = 0.000272/0.000564 = 0.482, which, as expected from Question 1, is not significant.

4. The sample PACF of the log returns shows some minor correlations at lags 1 and 2 so that an AR(2) model is employed. Write down the fitted AR(2) model, including the residual variance.

Answer: Let $\tilde{r}_t = r_t - 0.000272$. The model is $\tilde{r}_t = -0.083\tilde{r}_{t-1} - 0.071\tilde{r}_{t-2} + a_t$, where $\sigma_a^2 = 0.000471$.

5. Based on the output provided, does the log returns have conditional heteroscedasticity (ARCH-effect)? Why?

Answer: Yes, because $Q(10)$ of the squared residuals is 161.06 with $p$-value close to zero.

6. Suppose that the log return $r_t$ of an asset is normally distributed with mean 0.02 and standard error 0.4. What is the mean of the simple return of the asset?

Answer: $E(r_t) = \exp[0.02 + (0.4^2/2)] - 1 \approx 0.105$. 


7. Describe two methods for identifying the order of an AR time series.
   Answer: Information criteria and PACF.

   Answer: In-sample: use an information criterion. Out-sample: backtest with either RMSE or MAE.

9. (Questions 9 to 10) Suppose the daily log return $r_t$ of Stock A follows the model $r_t = 0.002 + a_t$, $a_t = \sigma_t \epsilon_t$, where $\{\epsilon_t\}$ is an independent and identically distributed (iid) sequence of standardized Student-$t$ distribution with 5 degrees of freedom. In addition, $\sigma_t^2 = 0.01 + 0.1 a_{t-1}^2 + 0.9 \sigma_{t-1}^2$. Let $h = 100$ be the forecast origin with $a_h = -0.015$ and $\sigma_h = 0.2$. Calculate the 1-step ahead prediction $r_h(1)$ and 1-step ahead volatility forecast.
   Answer: $r_h(1) = 0.002$. $\sigma_{h+1}^2 = 0.01 + 0.1(-0.015)^2 + 0.9(0.2)^2 = 0.046$ so that the 1-step ahead volatility forecast is $\sqrt{0.046} = 0.215$.

10. Calculate the 5-step ahead prediction $r_h(5)$ and the 5-step ahead volatility forecast at the forecast origin $h$.
    Answer: $r_h(5) = 0.002$. For volatility forecasts, use the IGARCH(1,1) model to obtain $\sigma_h^2(5) = 0.086$ so that $\sigma_h(5) = \sqrt{0.086} = 0.293$.

11. Consider a linear regression model. Describe a situation under which the conventional $R^2$ measure is not informative? Also, why is the commonly used Durbin-Watson statistic not sufficient in detecting serial correlations of time-series data?
    Answer: When both the dependent variable and the explanatory variable have unit roots. The DW-statistic is not sufficient because it only checks the lag-1 serial correlation of the residuals.

12. Let $r_t$ be a univariate time series. Consider the following two stationary AR(2) models: (a) $r_t = \phi_0 + \phi_2 r_{t-2} + a_t$ and (b) $r_t = \phi_0 + \phi_{21} r_{t-1} + \phi_{22} r_{t-2} + a_t$, where $a_t$ is a sequence of iid random variables with mean zero and variance $\sigma_a^2$. Why is $|\phi_2| < 1$ for model (a)? Is it possible that $\phi_{21} > 1$ in model (b)? Why?
    Answer: $|\phi_2| < 1$ because it is the correlation between $r_t$ and $r_{t-2}$. Yes, it is possible that $\phi_{21} > 1$. For example, $(1 - 1.3B + 0.4B^2)r_t = \phi_0 + a_t$ is a stationary AR(2) model.

13. Consider the univariate time series model $(1 - 0.9B + 0.2B^2)r_t = 100 + (1 - 1.1B)a_t$, where $a_t$ is a sequence of iid random variables with mean zero and variance $\sigma_a^2$. Is the model stationary? Why? Is the model invertible? Why?
Answer: The model is stationary because \((1 - 0.9B + 0.2B^2) = (1 - 0.5B)(1 - 0.4B)\). The model is not invertible because the MA(1) coefficient is greater than 1.

14. Give two univariate volatility models what can handle the leverage effect in asset returns?

Answer: Any two of EGARCH, Threshold GARCH, GJR, and APARCH models.

15. Consider the simple model \(r_t = 0.02 + a_t - 1.1a_{t-1} + 0.3a_{t-2}\), where \(a_t\) is defined in the prior question. Assume that \(\sigma_a^2 = 1\), \(a_{100} = -0.02\) and \(a_{99} = 0.01\). Calculate the 1-step and 3-step ahead point forecasts of \(r_t\) at the forecast origin \(t = 100\).

Answer: \(r_{100}(1) = 0.02 - 1.1(-0.02) + 0.3(0.01) = 0.045\). \(r_{100}(3) = 0.02\).

**Problem B.** (18 pts) Consider, again, the daily log returns of Qualcomm stock from May 17, 2007 to April 30, 2013. Several volatility models are fitted to the data and the relevant R output is attached. Answer the following questions.

1. (3 points) A volatility model, called \(m2\) in R, is entertained. Write down the fitted model, including the mean equation. Is the model adequate? Why?

Answer: \( \tilde{r}_t = -0.0267 \tilde{r}_{t-1} - 0.081 \tilde{r}_{t-2} + a_t\), \( a_t = \sigma_t \epsilon_t\), where \(\epsilon_t \sim N(0, 1)\) and \(\tilde{r}_t = r_t - 1.13 \times 10^{-3}\). The volatility model is \(\sigma_t^2 = 1.39 \times 10^{-5} + 0.0804 \sigma_{t-1}^2 + 0.892 \sigma_{t-1}^2\).

The model is not adequate because the normality of \(\epsilon_t\) is rejected.

2. (3 points) Another volatility model, called \(m3\) in R, is fitted to the returns. Write down the model, including all estimated parameters.

Answer: \( \tilde{r}_t = -0.0256 \tilde{r}_{t-1} - 0.0501 \tilde{r}_{t-2} + a_t\), \( \tilde{r}_t = r_t - 7.93 \times 10^{-4}\), \( a_t = \sigma_t \epsilon_t\) and \(\epsilon_t \sim t^{*}_{4.912}\). The volatility model is \(\sigma_t^2 = 5.72 \times 10^{-6} + 0.0731 \sigma_{t-1}^2 + 0.917 \sigma_{t-1}^2\).

3. (3 points) A third model, called \(m4\) in R, is also entertained. Write down the model, including the distributional parameters.

Answer: \( \tilde{r}_t = -0.021 \tilde{r}_{t-1} - 0.0467 \tilde{r}_{t-2} + a_t\), \( \tilde{r}_t = r_t - 1.014 \times 10^{-3}\), \( a_t = \sigma_t \epsilon_t\) with \(\epsilon_t \sim t^{*}_{4.912,1.05}\), which is a skew standardized Student \(t\) distribution with 4.912 degrees of freedom. The volatility model is \(\sigma_t^2 = 5.69 \times 10^{-6} + 0.0741 \sigma_{t-1}^2 + 0.917 \sigma_{t-1}^2\).
4. (3 points) Let $\xi$ be the skew parameter in model m4. Does the estimate of $\xi$ confirm that the distribution of the log returns is skewed? Why? Perform the test to support your answer.

Answer: $t$-ratio = $(1.05 - 1)/0.0384 = 1.30$, which is less than 1.96 so that the null hypothesis of $\xi = 1$ is not rejected, implying that the distribution of the log return is not skewed.

5. (2 points) Compare the three models m2, m3, m4. Which model is preferred? Why?

Answer: m3 is preferred. It has a smaller AIC compared with m2. It is preferred over m4 because the distribution of the log returns is not skewed.

6. (2 points) To facilitate Value at Risk calculation via the RiskMetrics approach, an IGARCH(1,1) model is fitted to the daily log returns. Write down the fitted IGARCH(1,1) model.

Answer: The model is $r_t = a_t$, $a_t = \sigma_t \epsilon_t$ with $\epsilon_t \sim N(0, 1)$ and the volatility equation is $\sigma_t^2 = (1 - 0.977)a_{t-1}^2 + 0.977\sigma_{t-1}^2$.

7. (2 points) Based on the output provided at $t = 1499$. Calculate the 1-step ahead volatility forecast using the IGARCH(1,1) model.

Answer: $\sigma_{1500}^2 = (1 - 0.977)(-0.000162)^2 + 0.977(0.01421)^2 = 0.0001973$ so that $\sigma_{1499}(1) = 0.014$.

Problem C. (17 points) Consider the monthly log return of 3M stock from January 1961 to December 2012. Use the attached R output to answer the following questions. Let $r_t$ denote the monthly log return.

1. (2 points) A simple GARCH(1,1) model is fitted to the data, called m1. Is the model adequate? If not, describe a method to improve the model.

Answer: No, the normality assumption is rejected.

2. (2 points) A GARCH(1,1) model is fitted to the data with Student $t$ innovations, called m2. Is the model adequate? Why?

Answer: Yes, all model checking statistics have $p$-values greater than 0.05.

3. (2 points) Based on the model m2, compute 95% 6-step ahead interval forecast for the 3M stock return at the forecast origin December 2012 (last data point).

Answer: The 95% interval forecast is $0.00806 \pm 1.96 \times 0.0578$. That is, $(-0.105, 0.121)$. 

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4. (3 points) To study the leverage effect, a TGARCH or GJR-type of model is entertained using the APARCH model in R. Write down the fitted model, called \( m_3 \).

Answer: The mean equation is \( r_t = 0.0071 + a_t, a_t = \sigma_t \epsilon_t, \epsilon_t \sim t_{8.11}^* \).

The volatility equation is \( \sigma_t^2 = 0.000658 + 0.0397(|a_{t-1}| + 1.0a_{t-1})^2 + 0.74\sigma_{t-1}^2 \). [You may use \(-1.0a_{t-1}\) as shown in the handout even though it is a typo.]

5. (3 points) Based on the fitted model \( m_3 \), is the leverage effect significant? State the null and alternative hypotheses, obtain the test statistic, and draw the conclusion.

Answer: \( H_0 : \gamma_1 = 0 \) versus \( H_a : \gamma_1 \neq 1 \). The \( t \)-ratio is zero so that we cannot reject \( H_0 \).

6. (3 points) A GARCH-M model is fitted to the percentage log returns, i.e. \( x_t = r_t \times 100 \). Write down the fitted model.

Answer: \( x_t = 0.739 + 0.0157\sigma_t^2 + a_t, a_t = \sigma_t \epsilon_t, \epsilon_t \sim N(0,1) \). The volatility equation is \( \sigma_t^2 = 7.719 + 0.0968a_{t-1}^2 + 0.695\sigma_{t-1}^2 \).

7. (2 points) Is the risk premium significantly different from zero? Why?

Answer: No, because its \( t \)-ratio is 0.096, which is less than 1.96.

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**Problem D. (20 points)** Consider the monthly unemployment rate, seasonally adjusted, of California from January 1976 to March 2013. Since the series is close to having a unit root, we focus on the first differenced series denoted by \( dca \) in the attached R analysis. To help predict the series, we use the lag-1 monthly unemployment rate of the U.S. as an explanatory variable. Denote the first differenced U.S. unemployment rate of \( dus \). Use the attached output to answer the following questions:

1. (3 points) A preliminary examination of the ACF and PACF suggests an ARIMA(3,0,1) model for the \( dca \) series. Write down the fitted model, including the residual variance.

   Answer: \( r_t = 1.006r_{t-1} + 0.25r_{t-2} - 0.328r_{t-3} + a_t - 0.454a_{t-1}, \sigma_a^2 = 0.00463 \).

2. (2 points) The fitted ARIMA(3,0,1) model does not contain a constant. Why?

   Answer: Because the t-test of the mean of \( dca \) has a \( p \)-value 0.972. Thus, the null hypothesis of zero mean cannot be rejected.
3. (3 points) Model checking shows that the lag-6 ACF of the residuals of the fitted ARIMA(3,0,1) model is significant. This led to an ARIMA(3,0,6) model, which contains several insignificant MA coefficients. Write down the refined ARIMA(3,0,6) model all estimates of which are statistically significant.

Answer: \( r_t = 0.519 r_{t-1} + 0.55 r_{t-2} - 0.209 r_{t-3} + a_t + 0.248 a_{t-3} - 0.134 a_{t-6}, \quad \sigma_a^2 = 0.00443. \)

4. (2 points) Provide a justification that it is ok to remove the insignificant parameters in the ARIMA(3,0,6) model. i.e., compare the full model (with all coefficients) with the refined model.

Answer: The AIC of the refined model is \(-1137.23\) which is smaller than the original ARMA(3,0,6) model.

5. (2 points) Does the fitted ARIMA(3,0,6) model for the monthly California unemployment rate imply the existence of business cycles? Why?

Answer: No, all three roots are real valued.

6. (3 points) Let \( y \) be the dca series and \( x \) be the lag-1 dus series. Write down the regression model with time series error between \( y \) and \( x \).

Answer: \( (1 - 0.487 B - 0.556 B^2 - 0.0079 B^3 + 0.185 B^4)(y_t - 0.0708 x_t) = a_t, \quad \sigma_a^2 = 0.00439. \)

7. (2 points) Based on the regression model with time series errors, is the lag-1 U.S. unemployment rate helpful in predicting the California unemployment rate? Why?

Answer: Yes, because the coefficient 0.0708 is significant at the 5% level.

8. (3 points) Compare the ARIMA(3,0,6) model and the regression model with time series errors. Which model is preferred? Why?

Answer: The regression model with time series error is preferred because it outperforms the ARIMA(3,0,6) model in out-sample forecasts. See the RMSE and MAE of backtest.

Problem E. (15 points) Consider the quarterly earnings per share of FedEx starting from the second quarter of 1992 to the fourth quarter of 2006. We analyze the log earnings per share, denoted by \( y_t \). Answer the following questions.

1. (3 points) Write down the fitted time series model \( m1 \) for the \( y_t \) series, including the residual variance.
2. (2 points) Is the model adequate? Why?
   Answer: Yes, because $Q(12)$ of the residuals is 15.29 with $p$-value greater than 5 percent.

3. (2 points) Write down the fitted time series model $m_2$ for $y_t$, including the residual variance.
   Answer: $(1 - B)(1 - B^4)y_t = (1 - 0.705B + 0.423B^2)(1 - 0.575B^4)a_t$, $\sigma^2 = 0.0636$.

4. (3 points) Let $\theta_3$ denote the coefficient of lag-3 of MA polynomial. Test $H_0 : \theta_3 = 0$ versus $H_a : \theta_3 \neq 0$. Calculate the test statistic and draw the conclusion.
   Answer: The $t$-ratio is $t = -0.3977/0.2064 = -1.927$. Since $|t| < 1.96$, the null hypothesis cannot be rejected at the 5% level. Thus, $\theta_3 = 0$.

5. (3 points) Write down the fitted time series model $m_3$ for $y_t$, including the residual variance.
   Answer: $(1 - B)(1 - B^4)(1 + 0.757B)y_t = (1 - 0.597B^4)a_t$, $\sigma^2 = 0.0591$.

6. (2 points) Among the three models $m_1$, $m_2$, $m_3$, which model is preferred? Why?
   Answer: $m_3$ is preferred because its residuals have no serial correlations and the model has the smallest AIC.