Key concept: Ito’s lemma

Stock Options:

- A contract giving its holder the right, but not obligation, to trade shares of a common stock by a certain date for a specified price. (In US, a contract involves 100 shares.)
- Call option: to buy
- Put option: to sell
- Specified price: strike price $K$
- Date: expiration $T$ (measured in years)

Note: You can also write a call or put option (underwrite).

Factors affecting the price of an option

- Current stock price: $P_t$
- Time to expiration: $T - t$
- Risk-free interest rate: $r$ per annum
- Stock volatility: $\sigma$ annualized

Payoff for European options (exercised at $T$ only)

Call option:

\[
V(P_T) = (P_T - K)_+ = \begin{cases} 
    P_T - K & \text{if } P_T > K \\
    0 & \text{if } P_T \leq K
\end{cases}
\]
The holder only exercises her option if \( P_T > K \) (buys the stock via exercising the option and sells the stock on the market).

**Put option:**

\[
V(P_T) = (K - P_T)_+ = \begin{cases} 
K - P_T & \text{if } P_T < K \\
0 & \text{if } P_T \geq K 
\end{cases}
\]

The holder only exercises her option if \( P_T < K \) (buys the stock from the market and sells it via option).

**Mathematical framework**

- Stock (log) price follows a diffusion equation, i.e. a continuous-time continuous stochastic process such as

\[
dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t,
\]

where \( \mu(x_t, t) \) and \( \sigma(x_t, t) \) are the drift and diffusion coefficient, respectively, and \( w_t \) is a standard Brownian motion (or Wiener process).

- In a complete market, use hedging to derive the price of an option (no arbitrage argument).

- In an incomplete market (e.g. existence of jumps), specify risk and a hedging strategy to minimize the risk.

**Stochastic processes**

- Wiener process (or Standard Brownian motion)
  - notation: \( w_t \)
  - initial value: \( w_0 = 0 \)
  - small increments are independent and normal
  - time points: \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \)
\{\Delta w_i = w_{t_i} - w_{t_{i-1}}\} \text{ are independent}

\Delta w_t = w_{t+\Delta t} - w_t \sim N(0, \Delta t).

- property: \(w_t \sim N(0, t)\)
- zero drift and rate of variance change is 1. That is,
  \[ dw_t = 0dt + 1dw_t. \]

- A simple way to understand Wiener processes is to do simulation. In R, this can be achieved by using:

\begin{verbatim}
  n=5000
  at = rnorm(n)
  wt = cumsum(at)/sqrt(n)
  plot(wt,type='l')
\end{verbatim}

Repeat the above commands to generate lots of “wt” series.

- Generalized Wiener process
  \[ dx_t = \mu dt + \sigma dw_t, \]
  where the drift \(\mu\) & rate of volatility change \(\sigma\) are constant.
- Ito’s process
  \[ dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t, \]
  where both drift and volatility are time-varying.
- Geometric Brownian motion
  \[ dP_t = \mu P_t dt + \sigma P_t dw_t, \]
  so that \(\mu(P_t, t) = \mu P_t\) and \(\sigma(P_t, t) = \sigma P_t\) with \(\mu\) and \(\sigma\) being constant.
Simulated Standard Brownian Motions

Figure 1: Time plots of four simulated Wiener processes
**Illustration:** Four simulated standard Brownian motions. Key feature: variability increases with time.

Assume that the price of a stock follows a geometric Brownian motion. What is the distribution of the log return?

To answer this question, we need Ito’s calculus.

**Review of differentiation**

$G(x)$: a differentiable function of $x$.

What is $dG(x)$?

Taylor expansion:

$$\Delta G \equiv G(x + \Delta x) - G(x) = \frac{\partial G}{\partial x} \Delta x$$

$$+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{1}{6} \frac{\partial^3 G}{\partial x^3} (\Delta x)^3 + \cdots.$$ 

Letting $\Delta x \to 0$, we have

$$dG = \frac{\partial G}{\partial x} dx.$$ 

How about $G(x, y)$?

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y$$

$$+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial y} \Delta x \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} (\Delta y)^2 + \cdots.$$ 

Taking limit as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy.$$ 

**Stochastic differentiation**

Now, consider $G(x_t, t)$ with $x_t$ being an Ito’s process.
\[ \Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t \]
\[ + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + \ldots . \]

A discretized version of the Ito’s process is
\[ \Delta x = \mu_* \Delta t + \sigma_* \epsilon \sqrt{\Delta t}, \]
where \( \mu_* = \mu(x_t, t) \) and \( \sigma_* = \sigma(x_t, t) \). Therefore,
\[ (\Delta x)^2 = \mu_*^2 (\Delta t)^2 + \sigma_*^2 \epsilon^2 \Delta t + 2 \mu_* \sigma_* \epsilon (\Delta t)^{3/2} \]
\[ = \sigma_*^2 \epsilon^2 \Delta t + H(\Delta t). \]

Thus, \((\Delta x)^2\) contains a term of order \(\Delta t\).
\[ E(\sigma_*^2 \epsilon^2 \Delta t) = \sigma_*^2 \Delta t, \]
\[ \text{Var}(\sigma_*^2 \epsilon^2 \Delta t) = E[\sigma_*^4 \epsilon^4 (\Delta t)^2] - [E(\sigma_*^2 \epsilon^2 \Delta t)]^2 = 2 \sigma_*^4 (\Delta t)^2, \]
where we use \( E(\epsilon^4) = 3 \). These two properties show that
\[ \sigma_*^2 \epsilon^2 \Delta t \to \sigma_*^2 \Delta t \quad \text{as} \quad \Delta t \to 0. \]

Consequently,
\[ (\Delta x)^2 \to \sigma_*^2 dt \quad \text{as} \quad \Delta t \to 0. \]

Using this result, we have
\[ dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma_*^2 dt \]
\[ = \left( \frac{\partial G}{\partial x} \mu_* + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma_*^2 \right) dt + \frac{\partial G}{\partial x} \sigma_* dw_t. \]

This is the well-known Ito’s lemma.
Example 1. Let $G(w_t, t) = w_t^2$. What is $dG(w_t, t)$?
Answer: Here $\mu_* = 0$ and $\sigma_* = 1$.

$$\frac{\partial G}{\partial w_t} = 2w_t, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{\partial^2 G}{\partial w_t^2} = 2.$$ 

Therefore,

$$dw_t^2 = (2w_t \times 0 + 0 + \frac{1}{2} \times 2 \times 1)dt + 2w_t dw_t = dt + 2w_t dw_t.$$ 

Example 2. If $P_t$ follows a geometric Brownian motion, what is the model for $\ln(P_t)$?
Answer: Let $G(P_t, t) = \ln(P_t)$. we have

$$\frac{\partial G}{\partial P_t} = \frac{1}{P_t}, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} = \frac{1-1}{2} \frac{P_t^2}{P_t^2}.$$ 

Consequently, via Ito’s lemma, we obtain

$$d\ln(P_t) = \left(\frac{1}{P_t} \mu P_t + \frac{1-1}{2} \frac{\sigma^2}{P_t^2} P_t^2\right) dt + \frac{1}{P_t} \sigma P_t dw_t$$

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dw_t.$$ 

Thus, $\ln(P_t)$ follows a generalized Wiener Process with drift rate $\mu - \sigma^2/2$ and variance rate $\sigma^2$.

The log return from $t$ to $T$ is normal with mean $(\mu - \sigma^2/2)(T - t)$ and variance $\sigma^2(T - t)$.

This result enables us to perform simulation. Let $\Delta t$ be the time interval. Then, the result says

$$\ln(P_{t+\Delta t}) = \ln(P_t) + (\mu - \sigma^2/2)\Delta t + \sqrt{\Delta t} \sigma \epsilon_{t+\Delta t},$$
where $\epsilon_{t+\Delta t}$ is a N(0,1) random variable. By generating $\epsilon_{t+\Delta t}$, we can obtain a simulated value for $\ln(P_{t+\Delta t})$. If one repeats the above simulation many times, one can take the average of $\ln(P_{t+\Delta t})$ as the expectation of $\ln(P_{t+\Delta t})$ given the model and $\ln(P_t)$. More specifically, one can use simulation to generate a distribution for $\ln(P_{t+\Delta t})$. In mathematical finance, this technique is referred to as discretization of a stochastic diffusion equation (SDE). The distribution of $\ln(P_{t+\Delta t})$ obtained in this way works when $P_t$ follows a geometric Brownian motion. The distribution, in general, is not equivalent to the true distribution of $\ln(P_{t+\Delta t})$ for a general SDE. Certain conditions must be met for the two distributions to be equivalent. Details are beyond this class. However, some results are available in the literature concerning the relationship between discrete-time distribution and continuous-time distribution. In other words, one derives conditions under which a discrete-time model converges to a continuous-time model when the time interval $\Delta t$ approaches zero. For the GARCH-type models, GARCH-M has a diffusion limit under some regularity conditions.

Return to the geometric Brownian motion. Even though we have a close-form solution for options pricing when $\sigma$ is constant. We can simulate $\ln(P_{t+\Delta t})$ when $\sigma_t$ is time-varying. For instance, $\sigma_t$ may follow a GARCH(1,1) model. For options pricing, under risk neural assumption, $\mu$ is replaced by the risk-free interest rate per annum. This is another application of the volatility models discussed in Chapter 3.

**Example 3.** If $P_t$ follows a geometric Brownian motion, what is the model for $\frac{1}{P_t}$?
Answer: Let $G(P_t) = 1/P_t$. By Ito’s lemma, the solution is

$$d\frac{1}{P_t} = (-\mu + \sigma^2)\frac{1}{P_t} dt - \sigma \frac{1}{P_t} dw_t,$$

which is again a geometric Brownian motion.

**Estimation of $\mu$ and $\sigma$**

Assume that $n$ log returns are available, say $\{r_t | t = 1, \cdots, n\}$.

**Statistical theory:**

Estimate the mean and variance by the sample mean and variance.

$$\bar{r} = \frac{\sum_{t=1}^{n} r_t}{n},$$

$$s_r^2 = \frac{1}{n-1} \sum_{t=1}^{n} (r_t - \bar{r})^2.$$

**Remember the length of time interval!**

Let $\Delta$ be the length of time intervals measured in years. Then, the distribution of $r_t$ is

$$r_t \sim N[(\mu - \sigma^2/2)\Delta, \sigma^2\Delta].$$

We obtain the estimates

$$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}},$$

$$\hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta} + \frac{s_r^2}{2\Delta}.$$

**Example.** Daily log returns of IBM stock in 1998.

The data show $\bar{r} = 0.002276$ and $s_r = 0.01915$.

Since $\Delta = 1/252$ year, we obtain that

$$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = 0.3040, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.6198.$$
Thus, the estimated expected return was 61.98% and the standard deviation was 30.4% per annum for IBM stock in 1998.

**Example.** Daily log returns of Cisco stock in 1999.
Data show $\bar{r} = 0.00332$ and $s_r = 0.026303$,
Also, $Q(12) = 10.8$. Therefore, we have
\[
\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = \frac{0.026303}{\sqrt{1.0/252.0}} = 0.418, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.924.
\]
Expected return was 92.4% per annum
Estimated s.d. was 41.8% per annum.

Data show $\bar{r} = -0.00301$ and $s_r = 0.05192$.
Therefore, $\hat{\sigma} = 0.818 \hat{\mu} = -0.412$.
Time-varying nature of mean and volatility is clearly shown.

**Distributions of stock prices**
If the price follows
\[dP_t = \mu P_t dt + \sigma P_t dw_t,\]
then,
\[
\ln(P_T) - \ln(P_t) \sim N \left[ (\mu - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t) \right].
\]
Consequently, given $P_t$,
\[
\ln(P_T) \sim N \left[ \ln(P_t) + (\mu - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t) \right],
\]
and we obtain (log-normal dist; ch. 1)
\[
E(P_T) = P_t \exp[\mu(T - t)],
\]
\[
\text{Var}(P_T) = P_t^2 \exp[2\mu(T - t)]\{\exp[\sigma^2(T - t)] - 1\}.
\]
The result can be used to make inference about $P_T$. Simulation is often used to study the behavior of $P_T$.

**Black-Scholes equation**

- Price of stock: $P_t$ is a Geo. B. Motion
- price of derivative: $G_t = G(P_t, t)$ contingent the stock
- Risk neutral world: expected returns are given by the risk-free interest rate (no arbitrage)

From Ito’s lemma:

$$dG_t = \left( \frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) dt + \frac{\partial G_t}{\partial P_t} \sigma P_t dw_t.$$

A discretized version of the set-up:

$$\Delta P_t = \mu P_t \Delta t + \sigma P_t \Delta w_t,$$

$$\Delta G_t = \left( \frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t + \frac{\partial G_t}{\partial P_t} \sigma P_t \Delta w_t,$$

Consider the **Portfolio**:

- short on derivative
- long $\frac{\partial G_t}{\partial P_t}$ shares of the stock.

Value of the portfolio is

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t.$$

The change in value is

$$\Delta V_t = -\Delta G_t + \frac{\partial G_t}{\partial P_t} \Delta P_t.$$
by substitution, we have

\[ \Delta V_t = \left( -\frac{\partial G_t}{\partial t} - \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t. \]

**No stochastic** component involved.

The portfolio must be risk-free during a small time interval.

\[ \Delta V_t = r V_t \Delta t \]

where \( r \) is the risk-free interest rate. We then have

\[ \left( \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t = r \left( G_t - \frac{\partial G_t}{\partial P_t} P_t \right) \Delta t. \]

and

\[ \frac{\partial G_t}{\partial t} + r P_t \frac{\partial G_t}{\partial P_t} + \frac{1}{2} \sigma^2 P_t^2 \frac{\partial^2 G_t}{\partial P_t^2} = r G_t, \]

the Black-Scholes differential equation for derivative pricing.

**Example.** A forward contract on a stock (no dividend). Here

\[ G_t = P_t - K \exp[-r(T - t)] \]

where \( K \) is the delivery price. We have

\[ \frac{\partial G_t}{\partial t} = -r K \exp[-r(T - t)], \quad \frac{\partial G_t}{\partial P_t} = 1, \quad \frac{\partial^2 G_t}{\partial P_t^2} = 0. \]

Substituting these quantities into LHS yields

\[ -r K \exp[-r(T - t)] + r P_t = r \{ P_t - K \exp[-r(T - t)] \}, \]

which equals RHS.

**Black-Scholes formulas**
A European call option: expected payoff

\[ E_\ast[\max(P_T - K, 0)] \]

Price of the call: (current value)

\[ c_t = \exp[-r(T - t)]E_\ast[\max(P_T - K, 0)]. \]

In a risk-neutral world, \( \mu = r \) so that

\[ \ln(P_T) \sim N \left[ \ln(P_t) + \left( r - \frac{\sigma^2}{2} \right)(T - t), \sigma^2(T - t) \right]. \]

Let \( g(P_T) \) be the pdf of \( P_T \). Then,

\[ c_t = \exp[-r(T - t)] \int_{K}^{\infty} (P_T - K)g(P_T)dP_T. \]

After some algebra (appendix)

\[ c_t = P_t\Phi(h_+) - K \exp[-r(T - t)]\Phi(h_-) \]

where \( \Phi(x) \) is the CDF of \( N(0,1) \),

\[ h_+ = \frac{\ln(P_t/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \]
\[ h_- = \frac{\ln(P_t/K) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} = h_+ - \sigma \sqrt{T - t}. \]

See Chapter 6 for some interpretations of the formula.

For put option:

\[ p_t = K \exp[-r(T - t)]\Phi(-h_-) - P_t\Phi(-h_+). \]

Alternatively, use the put-call parity:

\[ p_t - c_t = K \exp[-r(T - t)] - P_t. \]

**Put-call parity:** Same underlying stock, same strike price, same time to maturity.
Current amount: $c - p - P + Ke^{-r(T-t)}$.

On the expiration date:

1. If $P^* \leq K$: call is worthless, put is $K - P^*$, stock $P^*$ and owe bank $K$. Net worth = 0.

2. If $K < P^*$: call is $K - P^*$, put is worthless, stock $P^*$ and owe $K$. Net worth = 0.

Under no arbitrage assumption: The initial position should be zero. That is, $c = p + P - Ke^{-r(T-t)}$.

**Example.** $P_t = $80. $\sigma = 20\%$ per annum. $r = 8\%$ per annum. What is the price of a European call option with a strike price of $90 that will expire in 3 months?

From the assumptions, we have $P_t = 80$, $K = 90$, $T - t = 0.25$, $\sigma = 0.2$ and $r = 0.08$. Therefore,

$$h_+ = \frac{\ln(80/90) + (0.08 + 0.04/2) \times 0.25}{0.2 \sqrt{0.25}} = -0.9278$$

$$h_- = h_+ - 0.2 \sqrt{0.25} = -1.0278.$$  

It can be found

$$\Phi(-0.9278) = 0.1767, \quad \Phi(-1.0278) = 0.1520.$$
Therefore,
\[ c_t = $80\Phi(-0.9278) - $90\Phi(-1.0278) \exp(-0.02) = $0.73. \]
The stock price has to rise by $10.73 for the purchaser of the call option to break even.
If \( K = $81 \), then
\[ c_t = $80\Phi(0.125775) - $81 \exp(-0.02)\Phi(0.025775) = $3.49. \]

**Lower bounds of European options**: No dividends.
\[ c_t \geq P_t - K \exp[-r(T - t)]. \]
Why?
Consider two portfolios:
- **A**: One European call option plus cash \( K \exp[-r(T - t)] \).
- **B**: One share of the stock.
For A: Invest the cash at risk-free interest rate. At time \( T \), the value is \( K \). If \( P_T > K \), the call option is exercised so that the portfolio is worth \( P_T \). If \( P_T < K \), the call option expires at \( T \) and the portfolio is worth \( K \). Therefore, the value of the portfolio is \( \max(P_T, K) \).
For B: The value at time \( T \) is \( P_T \).
Thus, portfolio A must be worth more than portfolio B today; that is,
\[ c_t + K \exp[-r(T - t)] \geq P_t. \]
See Example 6.7 for an application.

**Stochastic integral**
The formula
\[ \int_0^t dx_s = x_t - x_0 \]
continues hold. In particular,
\[ \int_0^t dw_s = w_t - w_0 = w_t. \]

From
\[ dw_t^2 = dt + 2w_t dw_t \]
we have
\[ w_t^2 = t + 2 \int_0^t w_s dw_s. \]
Therefore,
\[ \int_0^t w_s dw_s = \frac{1}{2}(w_t^2 - t). \]
Different from \[ \int_0^t ydy = (y_t^2 - y_0^2)/2. \]
Assume \( x_t \) is a Geo. Brownian motion,
\[ dx_t = \mu x_t dt + \sigma x_t dw_t. \]
Apply Ito’s lemma to \( G(x_t, t) = \ln(x_t) \), we obtain
\[ d\ln(x_t) = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dw_t. \]
Taking integration, we have
\[ \int_0^t d\ln(x_s) = \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t ds + \sigma \int_0^t dw_s. \]
Consequently,
\[ \ln(x_t) = \ln(x_0) + (\mu - \sigma^2/2)t + \sigma w_t, \]
and
\[ x_t = x_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t]. \]
Change \( x_t \) to \( P_t \). The price is
\[ P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t]. \]

**Jump diffusion**
Weaknesses of diffusion models:
• no volatility smile (convex function of implied volatility vs strike price)
• fail to capture effects of rare events (tails)

Modification: jump diffusion and stochastic volatility

Jumps are governed by a probability law:

Poisson process: $X_t$ is a Poisson process if

$$Pr(X_t = m) = \frac{\lambda^m m^m}{m!} \exp(-\lambda t), \quad \lambda > 0.$$  

Use a special jump diffusion model by Kou (2002).

$$\frac{dP_t}{P_t} = \mu dt + \sigma dw_t + d \left( \sum_{i=1}^{n_t} (J_i - 1) \right),$$

• $w_t$: a Wiener process,
• $n_t$: a Poisson process with rate $\lambda$,
• $\{J_i\}$: iid such that $X = \ln(J)$ has a double exp. dist. with probability density function

$$f_X(x) = \frac{1}{2\eta} e^{-|x-\kappa|/\eta}, \quad 0 < \eta < 1.$$

• the above three processes are independent.

$n_t$ = the number of jumps in $[0, t]$ and Poisson($\lambda t$). At the $i$th jump, the proportion of price jump is $J_i - 1$.

For pdf of double exp. dist., see Figure 6.8 of the text.

Stock price under the jump diffusion model:

$$P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t] \prod_{i=1}^{n_t} J_i.$$
This result can be used to obtain the distribution for the return series.
Price of an option: Analytical results available, but complicated.

**Example** $P_t = $80. $K = $81. $r = 0.08$ and $T - t = 0.25$.
Jump: $\lambda = 10$, $\kappa = -0.02$ and $\eta = 0.02$.
We obtain $c_t = $3.92, which is higher than $3.49$ of Example 6.6.
$p_t = $3.31, which is also higher.

**Some greeks**: option value $V$, stock price $P$

1. Delta: $\Delta = \frac{\partial V}{\partial P}$
2. Gamma: $\Gamma = \frac{\partial \Delta}{\partial P}$
3. Theta: $\Theta = -\frac{\partial V}{\partial t}$.
4. Vega: partial derivative of $V$ with respect to volatility.