VaR of a Portfolio

**Basic setup:** Two assets with log returns \( r_{1t} \) and \( r_{2t} \). The portfolio consists of \( w_1 \) and \( w_2 \) amounts invested in asset 1 and asset 2, respectively.

Under RiskMetrics, we have

\[
\begin{align*}
  r_{1t} | F_{t-1} & \sim N(0, \sigma_{1t}^2), \quad \sigma_{1t}^2 = \beta \sigma_{1,t-1}^2 + (1 - \beta) r_{1,t-1}^2 \\
  r_{2t} | F_{t-1} & \sim N(0, \sigma_{2t}^2), \quad \sigma_{2t}^2 = \beta \sigma_{2,t-1}^2 + (1 - \beta) r_{2,t-1}^2.
\end{align*}
\]

VaR for the two assets are \( w_1 \text{VaR}_1 \) and \( w_2 \text{VaR}_2 \), respectively. For instance, for tail probability 0.05, we have VaR for the two assets as 1.645\( w_1 \sigma_{1t} \) and 1.645\( w_2 \sigma_{2t} \), respectively.

Let \( p_t \) be the log return of the portfolio. Then, we have

\[
p_t = w r_{1t} + (1 - w) r_{2t},
\]

where \( w = \frac{w_1}{w_1 + w_2} \) and \( 1 - w = \frac{w_2}{w_1 + w_2} \).

**Remark.** \((w_1 + w_2)w = w_1 \) and \((w_1 + w_2)(1 - w) = w_2\).

Under RiskMetrics, we have

\[
p_t | F_{t-1} \sim N(0, \sigma_{pt}^2),
\]

where

\[
\sigma_{pt}^2 = \text{Var}(r_{pt} | F_{t-1}) = w^2 \sigma_{1t}^2 + (1-w)^2 \sigma_{2t}^2 + 2w(1-w)\rho_t \sigma_{1t} \sigma_{2t}.
\]
The VaR for the portfolio is \((w_1 + w_2)\text{VaR}_p\). For tail probability of 0.05, we have \(\text{VaR}_p = 1.645(w_1 + w_2)\sigma_{pt}\). Therefore, the square of VaR for the portfolio with tail probability 0.05 is

\[
(VaR_p)^2 = (1.645)^2(w_1 + w_2)^2\sigma_{pt}^2
\]

\[
= (1.645)^2(w_1 + w_2)^2
\]

\[
\times [w^2\sigma_{1t}^2 + (1 - w)^2\sigma_{2t}^2 + 2w(1 - w)\rho_t\sigma_{1t}\sigma_{2t}]
\]

\[
= (1.645)^2[w_1^2\sigma_{1t}^2 + w_2^2\sigma_{2t}^2 + 2w_1w_2\rho_t\sigma_{1t}\sigma_{2t}]
\]

\[
= \text{VaR}_1^2 + \text{VaR}_2^2 + 2\rho_t \text{VAR}_1 \text{VaR}_2.
\]

This is exactly similar to

\[
\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\rho \times \text{std}(X)\text{std}(Y).
\]

The result can be generalized to more than two assets.