
(a) Is the expected value of $r_t$ zero? Why? Are there any serial correlations in $r_t$? Why?

Answer: The one sample $t$ test shows $t = 2.03$ with $p$-value 0.043 so that the mean is different from zero. The Ljung-Box statistic shows $Q(10) = 10.97$ with $p$-value 0.36. Therefore, there are no serial correlations.

(b) Fit a Gaussian ARMA-GARCH model to the $r_t$ series. Obtain the normal QQ-plot of the standardized residuals, and write down the fitted model. Is the model adequate? Why?

Answer: The QQ-plot is shown in Figure 1. The fitted model is

$$
\begin{align*}
    r_t &= 0.109 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \\
    \sigma_t^2 &= 0.023 + 0.0073 a_{t-1}^2 + 0.988 \sigma_{t-1}^2.
\end{align*}
$$

Based on the QQ-plot, the Gaussian assumption is rejected. The model is not adequate.

(c) Build an ARMA-GARCH model with Student-$t$ innovations for the $r_t$ series. Perform model checking and write down the fitted model.

Answer: Model checking indicates that the model is reasonable. The QQ-plot is in Figure 2. The fitted model is

$$
\begin{align*}
    r_t &= 0.087 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t^*_4 \\
    \sigma_t^2 &= 0.061 + 0.017 a_{t-1}^2 + 0.968 \sigma_{t-1}^2.
\end{align*}
$$

(d) Obtain 1-step to 5-step ahead mean and volatility forecasts using the fitted ARMA-GARCH model with Student-$t$ innovations.

Answer: The forecasts are

<table>
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<tr>
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<th>meanError</th>
<th>standardDeviation</th>
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2. Consider again the daily log returns, in percentages, of Amazon stock in Problem 1.

(a) Let $a_t = r_t - \bar{r}$, where $\bar{r}$ is the sample mean of $r_t$. Fit an IGARCH(1,1) model with a constant term in the volatility equation to the $a_t$ series. Write down the fitted model.

Answer: The model is

\begin{align*}
a_t &= \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \\
\sigma_t^2 &= 0.0014 + (1 - 0.9904)a_{t-1}^2 + 0.9904\sigma_{t-1}^2.
\end{align*}

(b) Let $\sigma_t$ be the fitted volatility of the IGARCH(1,1) model. Define the standardized residuals as $\epsilon_t = (r_t - \bar{r})/\sigma_t$. Is there any serial correlation in the standardized residuals? Why?

Answer: The Ljung-Box statistics show $Q(10) = 10.57$ with $p$-value 0.39, indicating that there are no serial correlations in the standardized residuals.

(c) Is there any serial correlation in the squares of the standardized residuals? Why?

Answer: The Ljung-Box statistics show $Q(10) = 3.88$ with $p$-value 0.95, indicating that there are no serial correlations in the squares of standardized residuals.

(d) Based on the model checking, is the IGARCH model adequate? Obtain 1-step to 4-step ahead volatility forecasts for the $r_t$ series (forecast origin is the last data point).

Answer: Use the fitted model to compute the 1-step ahead prediction. Then, add the drift term. The resulting volatility forecasts are 1.940, 1.940, 1.941, and 1.941.


Answer: The one-sample $t$ test shows $t = 4.35$ with $p$-value $1.59 \times 10^{-5}$, indicating that the mean is not zero. The Ljung-Box statistics of the returns show $Q(12) = 12.69$ with $p$-value 0.38. Therefore, there are no serial correlations in the returns. The Ljung-Box statistics of squared residuals show $Q(12) = 157.99$, which is highly significant. There are ARCH effects in the returns.

(b) Build a GARCH model with Gaussian innovations for the log return series. Perform model checking and write down the fitted model.

Answer: The fitted model is

\begin{align*}
r_t &= 0.012 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \\
\sigma_t^2 &= 4.96 \times 10^{-5} + 0.089a_{t-1}^2 + 0.902\sigma_{t-1}^2.
\end{align*}

Model checking shows that the Gaussian innovations are rejected.
(c) Fit an IGARCH(1,1) model for the MCD log returns. Write down the fitted model.
Answer: The IGARCH(1,1) model is
\[ r_t = 0.0116 + a_t, \quad a_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \]
\[ \sigma_t^2 = (1 - 0.926a_{t-1}^2 + 0.926\sigma_{t-1}^2). \]

(d) Fit a GARCH model with skew-Student-\(t\) innovations to the log return series. Perform model checking and write down the fitted model.
Answer: The fitted model is
\[ r_t = 0.012 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t^*_{10}(0.8333) \]
\[ \sigma_t^2 = 7.49 \times 10^{-5} + 0.097a_{t-1}^2 + 0.893\sigma_{t-1}^2. \]

Model checking shows that the model is adequate. The QQ-plot is in Figure 3.

(e) Based on the fitted model, is the monthly log returns of MCD stock skewed? Why?
Answer: Yes, the \(t\)-test for skew parameter is \(-3.29 = (0.8333 - 1)/0.0507\), which is less than \(-1.96\).

(f) Fit a GARCM-M model to the monthly log returns. Write down the model? Is the risk premium statistically significant? Why?
Answer: The fitted GARCH-M model is
\[ r_t = 0.004 + 1.66\sigma_t^2 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \]
\[ \sigma_t^2 = 4.85 \times 10^{-5} + 0.089a_{t-1}^2 + 0.902\sigma_{t-1}^2. \]

The risk premium is significant at the 5% level, because the \(t\)-ratio has a \(p\)-value of 0.047.

(g) Fit a TGARCH(1,1) model to the monthly log returns. Write down the fitted model. Is the leverage effect statistically significant? Why?
Answer: One can use the TGARCH(1,1) R script or the command \texttt{garchFit} with sub-command \texttt{leverage = T}. The fitted model is
\[ r_t = 0.0116 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \]
\[ \sigma_t^2 = 7.71 \times 10^{-5} + (0.096 + 0.129N_{t-1})a_{t-1}^2 + 0.889\sigma_{t-1}^2. \]

In this particular instance, the leverage effect is not significant at the 5% level, because its \(t\)-ratio has a \(p\)-value of 0.19. The conclusion does not change if one uses skew Student-\(t\) innovations. [See model \texttt{n7} in the R output.]

4. Monthly returns of the value-weighted index, including dividends from 1966 to 20014.
(a) Find an adequate model for the monthly log return series. Perform model checking to justify your model.
Answer: The mean of the returns is significant at the 5% level. Ljung-Box statistics show that $Q(12) = 10.87$ with $p$-value 0.54. Therefore, there are no serial correlations. The fitted model is

$$r_t = 0.00898 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t^*_{7.54}(0.743)$$

$$\sigma^2_t = 1.05 \times 10^{-4} + 0.109 a_{t-1}^2 + 0.842 \sigma^2_{t-1}.$$ 

Model checking indicates the model is adequate. See also the QQ plot in Figure 4. The skew parameter is highly significant. As a matter of fact, QQ plot of Student-t innovations indicates that skew distribution is needed. See Figure 5.

(b) Obtain 1-step to 5-step ahead predictions of the log return and its volatility at the forecast origin December 2014.
Answer: forecasts are

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<th>meanForecast</th>
<th>meanError</th>
<th>standardDeviation</th>
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<td>5 0.008977569 0.03492724</td>
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</table>

(c) Fit a GJR model (may use the APARCH command) to the monthly log return series. Write down the model. Is the leverage effect statistically significant? Why?
Answer: The fitted model is

$$r_t = 0.00811 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t^*_{7.64}(0.738)$$

$$\sigma^2_t = 1.93 \times 10^{-4} + 0.051(|a_{t-1}| - 0.981 a_{t-1})^2 + 0.802 \sigma^2_{t-1}.$$ 

No, the leverage parameter is not significant at the 5% level. It has a $p$-value of 0.23.


(a) Build a GARCH model (including mean equation) for the log return series. Perform model checking. Write down the fitted model.
Answer: The mean of the log return is not different from zero at the 5% level. However, the returns show some serial correlations. Based on the PACF, I started with mean equation being an ARMA(10,0). However, the estimation results show that only lag-6 AR coefficient is significant. A refined model is

$$r_t = 1.7 \times 10^{-5} - .0075 r_{t-1} + .0085 r_{t-2} - .0064 r_{t-3} - .018 r_{t-4} - .032 a_{t-5}$$
$$+ .064 r_{t-6} + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1)$$

$$\sigma^2_t = 1.2 \times 10^{-7} + 0.042 \sigma^2_{t-1} + 0.955 \sigma^2_{t-1}.$$
This model can handle the mean and volatility, but the normality assumption is rejected. To use Student-
$t$ innovations, one needs use percentage returns. Since the package does not allow for fixing AR coefficients to zero, the above model is shown.

(b) Let $r_t$ be the daily log return. For numeric stability, consider the percentage log return, i.e. $x_t = 100r_t$. Fit a volatility model with leverage effect to $x_t$. Write down the fitted model. Is the leverage effect statistically significant? Why?

Answer: Since the AR coefficients are small, I dropped them in this analysis. The fitted model is

$$
\begin{align*}
    r_t &= 0.00156 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_{10}^*, \\
    \sigma_t^2 &= 6.47 \times 10^{-4} + (0.021 + 0.422N_{t-1})a_{t-1}^2 + 0.971\sigma_{t-1}^2.
\end{align*}
$$

In this particular instance, the leverage effect is significant at the 5% level, because the $t$-ratio is 2.49 with $p$-value 0.013.
Figure 2: QQ-plot for the standardized residuals of GARCH(1,1) model with Student-$t$ innovations fitted to the daily log returns of Amazon stock.

Figure 3: QQ-plot for the standardized residuals of GARCH(1,1) model with skew Student-$t$ innovations fitted to the monthly log returns of McDonald’s stock.
Figure 4: QQ-plot for the standardized residuals of GARCH(1,1) model with skew Student-t innovations fitted to the monthly log returns of value-weighted index.

Figure 5: QQ-plot for the standardized residuals of GARCH(1,1) model with Student-t innovations fitted to the monthly log returns of value-weighted index.