What is asset volatility?
Answer: Conditional standard deviation of the asset returns

Why is volatility important?
Has many important applications

- Option (derivative) pricing, e.g., Black-Scholes formula
- Risk management, e.g. value at risk (VaR)
- Asset allocation, e.g., minimum-variance portfolio; see pages 184-185 of Campbell, Lo and MacKinlay (1997).
- Interval forecasts

A key characteristic: Not directly observable!!

How to calculate volatility?
There are several versions of volatility, but conditional standard deviation is commonly used.

1. Use high-frequency data: French, Schwert & Stambaugh (1987); see Section 3.15.
   - Realized volatility of daily log returns: use intraday high-frequency log returns.
   - Use daily high, low, and closing (log) prices, e.g. range = daily high - daily low.
2. Implied volatility of options data, e.g, VIX of CBOE. Figure 1.

3. Econometric modeling: use daily or monthly returns

We focus on the econometric modeling first. Use of high frequency data will be discussed later.

**Note:** In most applications, volatility is annualized. This can easily be done by taking care of the data frequency. For instance, if we use daily returns in econometric modeling, then the annualized volatility (in the U.S.) is

\[ \sigma^*_t = \sqrt{252}\sigma_t, \]

where \(\sigma_t\) is the estimated volatility derived from an employed model. If we use monthly returns, then the annualized volatility is

\[ \sigma^*_t = \sqrt{12}\sigma_t, \]

where \(\sigma_t\) is the estimated volatility derived from the employed model for the monthly returns. Our discussion, however, continues to use \(\sigma_t\) for simplicity.

**Basic idea** of econometric modeling:
Shocks of asset returns are **NOT** serially correlated, but dependent, implying that the serial dependence in asset returns is nonlinear. As shown by the ACF of returns and absolute returns of some assets we discussed so far.

**Basic structure**

\[ r_t = \mu_t + a_t, \quad \mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i}, \]

Volatility models are concerned with time-evolution of

\[ \sigma_t^2 = \text{Var}(r_t|F_{t-1}) = \text{Var}(a_t|F_{t-1}), \]
the conditional variance of the return $r_t$.

Consider the daily closing index of the S&P500 index from January 03, 2007 to April 14, 2016. The log returns follow approximately an MA(2) model

$$r_t = a_t - 0.109a_{t-1} - 0.054a_{t-2}, \quad \sigma^2 = 0.00018.$$  

**R Demonstration**

```r
> require(quantmod)
> getSymbols(~GSPC~, from="2007-01-03", to="2016-04-14")
[1] "GSPC"
> dim(GSPC)
[1] 2337 6
> head(GSPC)
GSPC.Open GSPC.High GSPC.Low GSPC.Close GSPC.Volume GSPC.Adjusted
2007-01-03 1418.03 1429.42 1407.86 1416.60 3429160000 1416.60
.. 2007-01-10 1408.70 1415.99 1405.32 1414.85 2764660000 1414.85
> spc = log(as.numeric(GSPC[,6]))
> rtn <- diff(spc)
```
Figure 2: Sample ACF of the squared residuals of an MA(2) model fitted to daily log returns of the S&P 500 index from January 3, 2007 to April 14, 2016.

```r
> acf(rtn)
> m1 <- arima(rtn, order=c(0,0,2))
> m1
Call: arima(x = rtn, order = c(0, 0, 2))

Coefficients:
    ma1    ma2 intercept
           -0.1093  -0.0541  2e-04
s.e.  0.0208  0.0215  2e-04

sigma^2 estimated as 0.0001813: log likelihood = 6748.04, aic = -13488.08
> m1 <- arima(rtn, order=c(0,0,2), include.mean=F)
> m1
Call: arima(x = rtn, order = c(0, 0, 2), include.mean = F)

Coefficients:
    ma1    ma2
           -0.1091  -0.0538
s.e.  0.0208  0.0215

sigma^2 estimated as 0.0001813: log likelihood = 6747.79, aic = -13489.58
> acf(m1$residuals)
```
> acf(m1$residuals^2)

How about the volatility?  
Is volatility constant over time?  
NO! See the ACF of squared residuals!  
How to model the evolving volatility?  

**Two general categories**  
- “Fixed function” and  
- Stochastic function  

of the available information.

**Univariate volatility models discussed:**  
1. Autoregressive conditional heteroscedastic (ARCH) model of Engle (1982),  
2. Generalized ARCH (GARCH) model of Bollerslev (1986),  
3. GARCH-M models,  
4. IGARCH models (used by RiskMetrics),  
5. Exponential GARCH (EGARCH) model of Nelson (1991),  
6. Threshold GARCH model of Zakoian (1994) or GJR model of Glosten, Jagannathan, and Runkle (1993),  
7. Asymmetric power ARCH (APARCH) models of Ding, Granger and Engle (1994), [TGARCH and GJR models are special cases of APARCH models.]  
8. Stochastic volatility (SV) models of Melino and Turnbull (1990), Harvey, Ruiz and Shephard (1994), and Jacquier, Polson and Rossi (1994).
ARCH model

\[ a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2, \]

where \( \{\epsilon_t\} \) is a sequence of iid r.v. with mean 0 and variance 1, \( \alpha_0 > 0 \) and \( \alpha_i \geq 0 \) for \( i > 0 \).

Distribution of \( \epsilon_t \): Standard normal, standardized Student-t, generalized error dist (ged), or their skewed counterparts.

Properties of ARCH models

Consider an ARCH(1) model

\[ a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2, \]

where \( \alpha_0 > 0 \) and \( \alpha_1 \geq 0 \).

1. \( \text{E}(a_t) = 0 \)

2. \( \text{Var}(a_t) = \alpha_0 / (1 - \alpha_1) \) if \( 0 < \alpha_1 < 1 \)

3. Under normality,

\[ m_4 = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)}, \]

provided \( 0 < \alpha_1^2 < 1/3 \).

The 3rd property implies heavy tails.

Advantages

- Simplicity
- Generates volatility clustering
- Heavy tails (high kurtosis)

Weaknesses
Symmetric between positive & negative prior returns

- Restrictive
- Provides no explanation
- Not sufficiently adaptive in prediction

**Building an ARCH Model**

1. Modeling the mean effect and testing for ARCH effects
   - \( H_0 \): no ARCH effects versus \( H_a \): ARCH effects
   - Use Q-statistics of squared residuals; McLeod and Li (1983) & Engle (1982)

2. Order determination
   - Use PACF of the squared residuals. (In practice, simply try some reasonable order).

3. Estimation: Conditional MLE

   - R provides many plots for model checking and for presenting the results.

5. Software: We use R with the package fGarch. (Other software available).

**Estimation:** Conditional MLE or Quasi MLE

**Special Note:** In this course, we estimate volatility models using the R package **fGarch** with **garchFit** command. The program is easy to use and allows for several types of innovational distributions:
The default is Gaussian (\textbf{norm}), standardized Student-\textit{t} distribution (\textbf{std}), generalized error distribution (\textbf{ged}), skew normal distribution (\textbf{snorm}), skew Student-\textit{t} (\textbf{sstd}), skew generalized error distribution (\textbf{sged}), and standardized inverse normal distribution (\textbf{snig}). Except for the inverse normal distribution, other distribution functions are discussed in the textbook. Readers should check the book for details about the density functions and their parameters.

\textbf{Example}: Monthly log returns of Intel stock

\textbf{R demonstration}: The \texttt{fGarch} package. Output edited.

```r
> library(fGarch)
> da=read.table("m-intc7303.txt",header=T)
> head(da)
     date      rtn
1 19730131 0.01005
... 6 19730629 0.13333
> intc=log(da$rtn+1)  # log returns
> acf(intc)
> acf(intc^2)
> pacf(intc^2)
> Box.test(intc^2,lag=10,type='Ljung')

Box-Ljung test

data: intc^2
X-squared = 59.7216, df = 10, p-value = 4.091e-09

> m1=garchFit(~garch(3,0),data=intc,trace=F)  # trace=F reduces the amount of output.
> summary(m1)

Title: GARCH Modelling
Call: garchFit(formula = ~garch(3, 0), data = intc, trace = F)

Mean and Variance Equation:
  data ~ garch(3, 0)
  [data = intc]

Conditional Distribution: norm

Coefficient(s):
    mu      omega   alpha1   alpha2   alpha3
0.016572  0.012043  0.208649  0.071837  0.049045
```
Std. Errors:
   based on Hessian

Error Analysis:

| Estimate  | Std. Error  | t value | Pr(>|t|) |
|-----------|-------------|---------|---------|
| mu        | 0.016572    | 0.006423| 2.580   | 0.00988  ** |
| omega     | 0.012043    | 0.001579| 7.627   | 2.4e-14  *** |
| alpha1    | 0.208649    | 0.129177| 1.615   | 0.10626  |
| alpha2    | 0.071837    | 0.048551| 1.480   | 0.13897  |
| alpha3    | 0.049045    | 0.048847| 1.004   | 0.31536  |

---

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R Chi^2 169.7731 0</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>R W 0.9606957 1.970413e-08</td>
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<tr>
<td>Ljung-Box Test R Q(10)</td>
<td>10.97025 0.3598405</td>
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<tr>
<td>Ljung-Box Test R Q(15)</td>
<td>19.59024 0.1882211</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(20)</td>
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<td>Ljung-Box Test R^2 Q(10)</td>
<td>5.376602 0.864644</td>
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<tr>
<td>Ljung-Box Test R^2 Q(15)</td>
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</tr>
<tr>
<td>Ljung-Box Test R^2 Q(20)</td>
<td>23.70577 0.255481</td>
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<tr>
<td>LM Arch Test R TR^2</td>
<td>20.48506 0.05844884</td>
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Information Criterion Statistics:

<table>
<thead>
<tr>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.228111</td>
<td>-1.175437</td>
<td>-1.228466</td>
<td>-1.207193</td>
</tr>
</tbody>
</table>

> m1=garchFit(~garch(1,0),data=intc,trace=F)
> summary(m1)

Title: GARCH Modelling

Call: garchFit(formula = ~garch(1, 0), data = intc, trace = F)

Mean and Variance Equation:

   data ~ garch(1, 0)
   [data = intc]

Conditional Distribution: norm

Coefficient(s):

   mu    omega   alpha1
0.016570 0.012490 0.363447

Std. Errors:
   based on Hessian
Error Analysis:

| Parameter | Estimate | Std. Error | t value | Pr(>|t|) |
|-----------|----------|------------|---------|----------|
| mu        | 0.016570 | 0.006161   | 2.689   | 0.00716  ** |
| omega     | 0.012490 | 0.001549   | 8.061   | 6.66e-16 *** |
| alpha1    | 0.363447 | 0.131598   | 2.762   | 0.00575  ** |

Log Likelihood:

230.2423  normalized: 0.6189309

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Test Type</th>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R^2 Chi^2</td>
<td>122.4040</td>
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<tr>
<td>Shapiro-Wilk Test</td>
<td>R W</td>
<td>0.9647629</td>
</tr>
<tr>
<td>Ljung-Box Test R</td>
<td>Q(10)</td>
<td>13.72604</td>
</tr>
<tr>
<td>Ljung-Box Test R</td>
<td>Q(15)</td>
<td>22.31714</td>
</tr>
<tr>
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<td>Q(20)</td>
<td>23.88257</td>
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<tr>
<td>Ljung-Box Test R^2</td>
<td>Q(20)</td>
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<tr>
<td>LM Arch Test R</td>
<td>TR^2</td>
<td>22.036</td>
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</table>

Information Criterion Statistics:

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<tr>
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<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
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</thead>
<tbody>
<tr>
<td>-1.221733</td>
<td>-1.190129</td>
<td>-1.221861</td>
<td>-1.209182</td>
</tr>
</tbody>
</table>

> plot(m1)

Make a plot selection (or 0 to exit):

1: Time Series
2: Conditional SD
3: Series with 2 Conditional SD Superimposed
4: ACF of Observations
5: ACF of Squared Observations
6: Cross Correlation
7: Residuals
8: Conditional SDs
9: Standardized Residuals
10: ACF of Standardized Residuals
11: ACF of Squared Standardized Residuals
12: Cross Correlation between r^2 and r
13: QQ-Plot of Standardized Residuals

Selection: 13

Make a plot selection (or 0 to exit):
The fitted ARCH(1) model is

\[ r_t = 0.0176 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \]
\[ \sigma_t^2 = 0.0125 + 0.363 \sigma_{t-1}^2. \]

Model checking statistics indicate that there are some higher order dependence in the volatility, e.g., see Q(15) for the squared standardized residuals. It turns out that a GARCH(1,1) model fares better for the data.

Next, consider Student-t innovations.

**R demonstration**

```r
> m2 = garchFit(~garch(1,0), data=intc, cond.dist="std", trace=F)
> summary(m2)
```

**Title:** GARCH Modelling

**Call:**

```
garchFit(formula = ~garch(1, 0), data = intc, cond.dist = "std",
          trace = F)
```

**Mean and Variance Equation:**

- `data ~ garch(1, 0)`
- `[data = intc]

**Conditional Distribution:**

\[ \text{std} \leq \text{Standardized Student-t.} \]
Figure 3: QQ-plot for standardized residuals of an ARCH(1) model with Gaussian innovations for monthly log returns of INTC stock: 1973 to 2003.

Coefficient(s):

\[
\begin{align*}
\text{mu} & \quad 0.021571 \\
\text{omega} & \quad 0.013424 \\
\text{alpha1} & \quad 0.259867 \\
\text{shape} & \quad 5.985979
\end{align*}
\]

Error Analysis:

\[
\begin{align*}
\text{mu} & \quad 0.021571 \quad 0.006054 \quad 3.563 \quad 0.000366 \quad *** \\
\text{omega} & \quad 0.013424 \quad 0.001968 \quad 6.820 \quad 9.09\text{e}-12 \quad *** \\
\text{alpha1} & \quad 0.259867 \quad 0.119901 \quad 2.167 \quad 0.030209 \quad * \\
\text{shape} & \quad 5.985979 \quad 1.660030 \quad 3.606 \quad 0.000311 \quad *** \quad <= \text{Estimate of degrees of freedom}
\end{align*}
\]

Log Likelihood:

242.9678 normalized: 0.6531391

Standardised Residuals Tests:

\[
\begin{align*}
\text{Statistic} & \quad \text{p-Value} \\
\text{Jarque-Bera Test} & \quad R \quad 130.8931 \quad 0 \\
\text{Shapiro-Wilk Test} & \quad R \quad W \quad 0.9637529 \quad 5.744026\text{e}-08 \\
\text{Ljung-Box Test} & \quad R \quad Q(10) \quad 14.31288 \quad 0.1591926 \\
\text{Ljung-Box Test} & \quad R \quad Q(15) \quad 23.34043 \quad 0.07717449 \\
\text{Ljung-Box Test} & \quad R \quad Q(20) \quad 24.87286 \quad 0.2063387
\end{align*}
\]
Ljung-Box Test  R^2  Q(10)  15.35917  0.1195054
Ljung-Box Test  R^2  Q(15)  33.96318  0.003446127
Ljung-Box Test  R^2  Q(20)  35.46828  0.01774746
LM Arch Test  R  TR^2  24.11517  0.01961957

Information Criterion Statistics:

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<thead>
<tr>
<th></th>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>-1.284773</td>
<td>-1.242634</td>
<td>-1.285001</td>
<td>-1.268039</td>
</tr>
</tbody>
</table>

> plot(m2)

Make a plot selection (or 0 to exit):

1: Time Series
2: Conditional SD
3: Series with 2 Conditional SD Superimposed
4: ACF of Observations
5: ACF of Squared Observations
6: Cross Correlation
7: Residuals
8: Conditional SDs
9: Standardized Residuals
10: ACF of Standardized Residuals
11: ACF of Squared Standardized Residuals
12: Cross Correlation between r^2 and r
13: QQ-Plot of Standardized Residuals

Selection: 13  <= The plot shows that the model needs further improvements.

> predict(m2,5)  <= Prediction

<table>
<thead>
<tr>
<th></th>
<th>meanForecast</th>
<th>meanError</th>
<th>standardDeviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.02157100</td>
<td>0.1207911</td>
<td>0.1207911</td>
</tr>
<tr>
<td>2</td>
<td>0.02157100</td>
<td>0.1312069</td>
<td>0.1312069</td>
</tr>
<tr>
<td>3</td>
<td>0.02157100</td>
<td>0.1337810</td>
<td>0.1337810</td>
</tr>
<tr>
<td>4</td>
<td>0.02157100</td>
<td>0.1344418</td>
<td>0.1344418</td>
</tr>
<tr>
<td>5</td>
<td>0.02157100</td>
<td>0.1346130</td>
<td>0.1346130</td>
</tr>
</tbody>
</table>

The fitted model with Student-t innovations is

\[ r_t = 0.0216 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon \sim t_{5.99} \]
\[ \sigma_t^2 = 0.0134 + 0.260a_{t-1}^2. \]

We use \( t_{5.99} \) to denote the standardized Student-t distribution with 5.99 d.f.

Comparison with normal innovations:
• Using a heavy-tailed dist for $\epsilon_t$ reduces the ARCH effect.
• The difference between the models is small for this particular instance.

You may try other distributions for $\epsilon_t$.

**GARCH Model**

$$a_t = \sigma_t \epsilon_t,$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

where $\{\epsilon_t\}$ is defined as before, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1$.

Re-parameterization:
Let $\eta_t = a_t^2 - \sigma_t^2$. $\{\eta_t\}$ un-correlated series.

The GARCH model becomes

$$a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^{s} \beta_j \eta_{t-j}.$$

This is an ARMA form for the squared series $a_t^2$.

Use it to understand properties of GARCH models, e.g. moment equations, forecasting, etc.

Focus on a GARCH(1,1) model

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

• Weak stationarity: $0 \leq \alpha_1, \beta_1 \leq 1, (\alpha_1 + \beta_1) < 1$.

• Volatility clusters

• Heavy tails: if $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$, then

$$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3.$$
• For 1-step ahead forecast,

\[ \sigma^2_h(1) = \alpha_0 + \alpha_1 a^2_h + \beta_1 \sigma^2_h. \]

For multi-step ahead forecasts, use \( a^2_t = \sigma^2_t \epsilon^2_t \) and rewrite the model as

\[ \sigma^2_{t+1} = \alpha_0 + (\alpha_1 + \beta_1) \sigma^2_t + \alpha_1 \sigma^2_t(\epsilon^2_t - 1). \]

2-step ahead volatility forecast

\[ \sigma^2_h(2) = \alpha_0 + (\alpha_1 + \beta_1) \sigma^2_h(1). \]

In general, we have

\[ \sigma^2_h(\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma^2_h(\ell - 1), \quad \ell > 1. \]

This result is exactly the same as that of an ARMA(1,1) model with AR polynomial \( 1 - (\alpha_1 + \beta_1)B \).

**Example:** Monthly excess returns of S&P 500 index starting from 1926 for 792 observations.

The fitted of a Gaussian AR(3) model

\[ \tilde{r}_t = r_t - 0.0062 \]
\[ \tilde{r}_t = .089\tilde{r}_{t-1} - .024\tilde{r}_{t-2} - .123\tilde{r}_{t-3} + .007 + a_t, \]
\[ \hat{\sigma}^2_a = 0.00333. \]

For the GARCH effects, use a GARCH(1,1) model, we have

A joint estimation:

\[ r_t = 0.032r_{t-1} - 0.030r_{t-2} - 0.011r_{t-3} + 0.0077 + a_t \]
\[ \sigma^2_t = 7.98 \times 10^{-5} + .853\sigma^2_{t-1} + 0.124a^2_{t-1}. \]
Implied unconditional variance of $a_t$ is
\[
\frac{0.0000798}{1 - 0.853 - 0.1243} = 0.00352
\]
close to the expected value. All AR coefficients are statistically insignificant.

A simplified model:
\[
r_t = 0.00745 + a_t, \sigma^2_t = 8.06 \times 10^{-5} + .854\sigma^2_{t-1} + .122a^2_{t-1}.
\]

Model checking:
For $\tilde{a}_t$: $Q(10) = 11.22(0.34)$ and $Q(20) = 24.30(0.23)$.
For $\tilde{a}^2_t$: $Q(10) = 9.92(0.45)$ and $Q(20) = 16.75(0.67)$.

Forecast: 1-step ahead forecast:
\[
\sigma^2_h(1) = 0.00008 + 0.854\sigma^2_h + 0.122a^2_h
\]

<table>
<thead>
<tr>
<th>Horizon</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>.054</td>
<td>.054</td>
<td>.054</td>
<td>.054</td>
<td>.059</td>
</tr>
</tbody>
</table>

**R demonstration:**

```R
> sp5=scan("sp500.txt")
Read 792 items
> pacf(sp5)
> m1=arima(sp5,order=c(3,0,0))
> m1
Call: arima(x = sp5, order = c(3, 0, 0))

Coefficients:
    ar1  ar2  ar3 intercept
ar1     0.0890 -0.0238 -0.1229     0.0062
s.e.     0.0353  0.0355  0.0353   0.0019

sigma^2 estimated as 0.00333:  log likelihood = 1135.25, aic=-2260.5
> m2=garchFit(~arma(3,0)+garch(1,1),data=sp5,trace=F)
> summary(m2)
Title:  GARCH Modelling
```

16
Call:
garchFit(formula = ~arma(3, 0) + garch(1, 1), data = sp5, trace = F)

Mean and Variance Equation:
data ~ arma(3, 0) + garch(1, 1)
[data = sp5]

Conditional Distribution: norm

Error Analysis:

| Estimate  | Std. Error | t value | Pr(>|t|) |
|-----------|------------|---------|----------|
| mu        | 7.708e-03  | 1.607e-03 | 4.798 | 1.61e-06 *** |
| ar1       | 3.197e-02  | 3.837e-02 | 0.833 | 0.40473 |
| ar2       | -3.026e-02 | 3.841e-02 | -0.788 | 0.43076 |
| ar3       | -1.066e-02 | 3.756e-02 | -0.284 | 0.77677 |
| omega     | 7.975e-05  | 2.810e-05 | 2.838 | 0.00454 ** |
| alpha1    | 1.242e-01  | 2.247e-02 | 5.529 | 3.22e-08 *** |
| beta1     | 8.530e-01  | 2.183e-02 | 39.075 | < 2e-16 *** |

---

Log Likelihood:
1272.179 normalized: 1.606287

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Test</th>
<th>Statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test R</td>
<td>Chi^2</td>
<td>73.04842</td>
</tr>
<tr>
<td>Shapiro-Wilk Test R</td>
<td>W</td>
<td>0.985797</td>
</tr>
<tr>
<td>Ljung-Box Test R</td>
<td>Q(10)</td>
<td>11.56744</td>
</tr>
<tr>
<td>Ljung-Box Test R</td>
<td>Q(15)</td>
<td>17.78747</td>
</tr>
<tr>
<td>Ljung-Box Test R</td>
<td>Q(20)</td>
<td>24.11916</td>
</tr>
<tr>
<td>Ljung-Box Test R^2</td>
<td>Q(10)</td>
<td>10.31614</td>
</tr>
<tr>
<td>Ljung-Box Test R^2</td>
<td>Q(15)</td>
<td>14.22819</td>
</tr>
<tr>
<td>Ljung-Box Test R^2</td>
<td>Q(20)</td>
<td>16.79404</td>
</tr>
<tr>
<td>LM Arch Test R</td>
<td>TR^2</td>
<td>13.34305</td>
</tr>
</tbody>
</table>

Information Criterion Statistics:

<table>
<thead>
<tr>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.194897</td>
<td>-3.153581</td>
<td>-3.195051</td>
<td>-3.179018</td>
</tr>
</tbody>
</table>

> m2 = garchFit(~garch(1, 1), data = sp5, trace = F)
> summary(m2)

Title: GARCH Modelling
Call: garchFit(formula = ~garch(1, 1), data = sp5, trace = F)

Mean and Variance Equation:
data ~ garch(1, 1)
[data = sp5]
Conditional Distribution: norm

Error Analysis:

|        | Estimate | Std. Error | t value | Pr(>|t|) |
|--------|----------|------------|---------|----------|
| mu     | 7.450e-03| 1.538e-03  | 4.845   | 1.27e-06 *** |
| omega  | 8.061e-05| 2.833e-05  | 2.845   | 0.00444 **  |
| alpha1 | 1.220e-01| 2.202e-02  | 5.540   | 3.02e-08 *** |
| beta1  | 8.544e-01| 2.175e-02  | 39.276  | < 2e-16 *** |

---

Log Likelihood:
1269.455 normalized: 1.602848

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Test</th>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test R</td>
<td>Chi^2</td>
<td>0</td>
</tr>
<tr>
<td>Shapiro-Wilk Test R W</td>
<td>0.9850517</td>
<td>3.141228e-07</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(10)</td>
<td>11.22050</td>
<td>0.340599</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(15)</td>
<td>17.99703</td>
<td>0.262822</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(20)</td>
<td>24.29896</td>
<td>0.229578</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(10)</td>
<td>9.920157</td>
<td>0.4475259</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(15)</td>
<td>14.21124</td>
<td>0.509572</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(20)</td>
<td>16.75081</td>
<td>0.6690903</td>
</tr>
<tr>
<td>LM Arch Test R TR^2</td>
<td>13.04872</td>
<td>0.3655092</td>
</tr>
</tbody>
</table>

Information Criterion Statistics:

<table>
<thead>
<tr>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
</thead>
</table>

> plot(m2)
Make a plot selection (or 0 to exit):

1: Time Series
2: Conditional SD
3: Series with 2 Conditional SD Superimposed
4: ACF of Observations
5: ACF of Squared Observations
6: Cross Correlation
7: Residuals
8: Conditional SDs
9: Standardized Residuals
10: ACF of Standardized Residuals
11: ACF of Squared Standardized Residuals
12: Cross Correlation between r^2 and r
13: QQ-Plot of Standardized Residuals
Selection: 3

> predict(m2,6)

<table>
<thead>
<tr>
<th></th>
<th>meanForecast</th>
<th>meanError</th>
<th>standardDeviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.007449721</td>
<td>0.05377242</td>
<td>0.05377242</td>
</tr>
<tr>
<td>2</td>
<td>0.007449721</td>
<td>0.05388567</td>
<td>0.05388567</td>
</tr>
<tr>
<td>3</td>
<td>0.007449721</td>
<td>0.05399601</td>
<td>0.05399601</td>
</tr>
<tr>
<td>4</td>
<td>0.007449721</td>
<td>0.05410353</td>
<td>0.05410353</td>
</tr>
<tr>
<td>5</td>
<td>0.007449721</td>
<td>0.05420829</td>
<td>0.05420829</td>
</tr>
<tr>
<td>6</td>
<td>0.007449721</td>
<td>0.05431038</td>
<td>0.05431038</td>
</tr>
</tbody>
</table>

Turn to Student-\(t\) innovation. (R output omitted.)

Estimation of degrees of freedom:

\[
r_t = 0.0085 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_\nu
\]

\[
\sigma_t^2 = 0.000125 + 0.113 a_{t-1}^2 + 0.842 \sigma_{t-1}^2,
\]

where the estimated degrees of freedom is 7.00.

Forecasting evaluation
Not easy to do; see Andersen and Bollerslev (1998).

**IGARCH model**

An IGARCH(1,1) model:

\[ a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) a_{t-1}^2. \]

For the monthly excess returns of the S&P 500 index, we have

\[ r_t = .007 + a_t, \sigma_t^2 = .0001 + .806 \sigma_{t-1}^2 + .194 a_{t-1}^2 \]

For an IGARCH(1,1) model,

\[ \sigma_h^2(\ell) = \sigma_h^2(1) + (\ell - 1) \alpha_0, \quad \ell \geq 1, \]

where \( h \) is the forecast origin.

Effect of \( \sigma_h^2(1) \) on future volatilities is persistent, and the volatility forecasts form a straight line with slope \( \alpha_0 \). See Nelson (1990) for more info.

Special case: \( \alpha_0 = 0 \). Volatility forecasts become a constant.

This property is used in RiskMetrics to VaR calculation.

**Example**: An IGARCH(1,1) model for the monthly excess returns of S&P500 index from 1926 to 1991 is given below via R.

\[ r_t = 0.0069 + a_t, \quad a_t = \sigma_t \epsilon_t \]

\[ \sigma_t^2 = 0.099 a_{t-1}^2 + .901 \sigma_{t-1}^2. \]

**R demonstration**: Using R script `Igarch.R`.

```r
> source("Igarch.R")
> sp5=scan(file="sp500.txt")
> Igarch(sp5,include.mean=T)
Estimates: 0.006874402 0.9007153
Maximized log-likehood: -1258.219

Coefficient(s):
              Estimate Std. Error  t value   Pr(>|t|)
mu       0.00687440   0.0015402 4.463323  8.07e-06 ***
beta     0.90071530   0.0158018 57.000822  < 2e-16 ***
```

20
Another R package: rugarch can be used to fit volatility models too.

```r
> sp5=scan("sp500.txt")
> require(rugarch)
> spec1=ugarchspec(variance.model=list(model="iGARCH",garchOrder=c(1,1)),
                     mean.model=list(armaOrder=c(0,0)))
> mm=ugarchfit(data=sp5,spec=spec1)
> mm

*---------------------------------*
* GARCH Model Fit              *
*---------------------------------*
Conditional Variance Dynamics
-----------------------------------
GARCH Model : iGARCH(1,1)
Mean Model : ARFIMA(0,0,0)
Distribution : norm

Optimal Parameters
-----------------------------------
     Estimate   Std. Error    t value  Pr(>|t|)
mu   0.0074175   0.0015253    4.86213   0.000001
omega 0.0000513   0.0000178   2.92377   0.003458
alpha1 0.1429510   0.0214437   6.66667   0.000000
beta1 0.8570490      NA       NA       NA

Robust Standard Errors:
-----------------------------------
     Estimate   Std. Error    t value  Pr(>|t|)
mu   0.0074175   0.0015867    4.67263   0.000003
omega 0.0000513   0.0000190   2.69128   0.007118
alpha1 0.1429510   0.0249777   5.72300   0.000000
beta1 0.8570490      NA       NA       NA

LogLikelihood:  1268.238

Information Criteria
-----------------------------------
Akaike   -3.1950
Bayes    -3.1773
Shibata  -3.1951
Hannan-Quinn -3.1882

Weighted Ljung-Box Test on Standardized Residuals
-----------------------------------
statistic   p-value
Lag[1]       0.5265   0.4681
Lag[2*(p+q)+(p+q)-1][2] 0.5304   0.6795
```
Lag[4*(p+q)+(p+q)-1][5] 2.5233 0.5009
d.o.f=0
H0: No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals

<table>
<thead>
<tr>
<th></th>
<th>statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lag[1]</td>
<td>1.166</td>
<td>0.2803</td>
</tr>
<tr>
<td>Lag[2*(p+q)+(p+q)-1][5]</td>
<td>2.672</td>
<td>0.4702</td>
</tr>
<tr>
<td>Lag[4*(p+q)+(p+q)-1][9]</td>
<td>4.506</td>
<td>0.5054</td>
</tr>
<tr>
<td>d.o.f=2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

.....