The solutions are brief. In some cases, there are multiple solutions and I can give a few that are most common. The notation in lectures and lecture notes is used throughout.

**Problem A**: (60 pts) Let \( \{a_t\} \) be a sequence of independent and identically distributed random variables with mean zero and variance \( \sigma^2_a \), \( \{Z_t\} \) be a time series, and \( B \) the back-shift operator such that \( BZ_t = Z_{t-1} \). Also all forecasts are based on the minimum mean squared error criterion. Briefly answer the following questions.

1. Suppose that \( Z_t = Z_{t-1} + a_t \). Define an innovational outlier (IO) for the series at time index \( t_0 \).

\[
Y_t = \frac{1}{1-B} (a_t + \omega I_t^{(t_0)}) = Z_t + \frac{\omega}{1-B} I_t^{(t_0)}.
\]

2. What is the impact of the IO on the \( Z_t \) process of Problem 1?

A: A level shift starting at time index \( t_0 \).

3. State two differences between intervention analysis and outlier analysis in a time series model.

A: (1) The time index of intervention is known. (2) The limiting distributions of the test statistics used are different.

4. Consider the ARMA(2,1) model \( (1-B + .2B^2)Z_t = (1 - 0.5B)a_t \). Put the model into a state-space form.

A: I give three forms, but there are other possibilities:

(a) Akaike’s approach

\[
\begin{bmatrix}
Z_{t+1} \\
Z_{t+2} \\
Z_{t+3}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -2 & 1
\end{bmatrix}
\begin{bmatrix}
Z_t \\
Z_{t+1} \\
Z_{t+2}
\end{bmatrix}
+ \begin{bmatrix}
1 \\
.5 \\
.3
\end{bmatrix} a_{t+1}
\]

\[
Z_t = [1, 0, 0] S_t.
\]

(b) Aoki’s approach:

\[
\begin{bmatrix}
Z_t \\
Z_{t-1} \\
a_t
\end{bmatrix}
= \begin{bmatrix}
1 & -2 & -5 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
z_{t-1} \\
z_{t-2} \\
a_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} a_t,
\]

\[
Z_t = [1, -.2, -.5] S_t + a_t.
\]

(c) Harvey’s approach:

\[
S_{t+1}
= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -2 & 1
\end{bmatrix}
S_t + \begin{bmatrix}
.5 \\
.3 \\
.2
\end{bmatrix} a_t
\]

1
5. Consider the model $Z_t = T_t + R_t$, where $T_t = T_{t-1} + e_t$ and $(1 - \phi B)R_t = (1 - \theta B)a_t$, where $\phi \neq \theta$ and \{\(e_t\)\} and \{\(a_t\)\} are two independent white noise processes. Put the model in a state-space form.

A: Again, I give two forms, but there are other possibilities:

(a) Akaike’s approach:

$$
\begin{bmatrix}
    T_{t+1} \\
    R_{t+1} \\
    R_{t+2|t+1}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & \phi
\end{bmatrix} \begin{bmatrix}
    T_t \\
    R_t \\
    R_{t+1|t}
\end{bmatrix} + \begin{bmatrix}
    1 & 0 \\
    0 & 1 \\
    0 & \phi - \theta
\end{bmatrix} \begin{bmatrix}
    e_{t+1} \\
    a_{t+1}
\end{bmatrix},
$$

$$
Z_t = [1, 1, 0]S_t.
$$

(b) Aoki’s approach:

$$
\begin{bmatrix}
    T_t \\
    R_t \\
    a_t
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & \phi & -\theta \\
    0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
    T_t \\
    R_t \\
    a_t
\end{bmatrix} + \begin{bmatrix}
    1 & 0 \\
    0 & 1 \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    e_t \\
    a_t
\end{bmatrix},
$$

$$
Z_t = [1, 1, -\theta]S_t + [1, 1] \begin{bmatrix}
    e_t \\
    a_t
\end{bmatrix}.
$$

6. Suppose that $Z_t$ follows a stationary AR(2) model $(1 - \phi_1 B - \phi_2 B^2)Z_t = a_t$ and that $Z_{100}$ is missing. Based on the model and the data \{\(Z_1, \ldots, Z_{150}\)\}. Provide a least squares estimate of $Z_{100}$.

A: Three observations involve $Z_{100}$. They are

$$
Z_{100} = \phi_1 Z_{99} + \phi_2 Z_{98} + a_{100},
$$

$$
Z_{101} = \phi_1 Z_{100} + \phi_2 Z_{99} + a_{101},
$$

$$
Z_{102} = \phi_1 Z_{101} + \phi_2 Z_{100} + a_{102}.
$$

Consequently, we obtain

$$
y_1 = 1Z_{100} + b_1, \quad y_1 = \phi_1 Z_{99} + \phi_2 Z_{98}, \quad b_1 = -a_{100}
$$

$$
y_2 = \phi_1 Z_{100} + b_2, \quad y_2 = Z_{101} - \phi_2 Z_{99}, \quad b_2 = -a_{101}
$$

$$
y_3 = \phi_2 Z_{100} + b_3, \quad y_3 = Z_{102} - \phi_1 Z_{101}, \quad b_3 = a_{102}.
$$

The OLS estimate of $Z_{100}$ is

$$
\hat{Z}_{100} = \frac{y_1 + \phi_1 y_2 + \phi_2 y_3}{1 + \phi_1^2 + \phi_2^2}.
$$

7. Suppose $Z_t = X_t + Y_t$, where $X_t$ is an AR(4) process and $Y_t$ is an ARMA(1,1) process. Assume further that $X_t$ and $Y_t$ are independent. What is the model for $Z_t$? It suffices to provide the maximum order.

A: An ARMA(5,5) model.
8. For an ARIMA process \( Z_t \), what is the relationship between the two forecasts \( Z_T(\ell) \) and \( Z_{T+1}(\ell-1) \), where \( Z_h(j) \) denotes the \( j \)-step ahead forecast of \( Z_{h+j} \) at the forecast origin \( h \).

\[
Z_{T+1}(\ell-1) = Z_T(\ell) + \psi_{\ell-1}a_{T+1},
\]

where \( \psi_i \) is the \( i \)th \( \psi \)-weight of \( Z_t \).

9. Suppose that \( Z_t \) follows the random-walk model

\[
Z_t = Z_{t-1} + a_t, \quad Z_0 = 0
\]

where, for simplicity, \( \{a_t\} \) is a sequence of independent and identically distributed normal random variables with mean zero and variance 1. Consider the linear regression

\[
\Delta Z_t = \beta Z_{t-1} + e_t,
\]

with the ordinary least squares estimate

\[
\hat{\beta} = \frac{\sum_{t=1}^{T} Z_{t-1}\Delta Z_t}{\sum_{t=1}^{T} Z_{t-1}},
\]

where \( \Delta Z_t = Z_t - Z_{t-1} \). What is the limiting distribution of \( T^{-1} \sum_{t=1}^{T} Z_{t-1}\Delta Z_t \) as \( T \to \infty \)? (No proof is needed, but keep it as simple as possible.)

\[
T^{-1} \sum_{t=1}^{T} Z_{t-1}\Delta Z_t \to_D \frac{1}{2} [W(1)^2 - 1],
\]

where \( W(r) \) is a standard Brownian motion (or Wiener process).

10. Consider an intervention model

\[
Y_t = \left( \omega_1 B + \frac{\omega_2 B^2}{1 - \delta B} \right) Z_t(h) + Z_t,
\]

where \( 0 < \delta < 1 \) and \( h \) is the time of intervention. What is the meaning of the parameter \( \omega_1 \)? Does the intervention have a permanent impact?

A: (1) \( \omega_1 \) denotes a delay (by 1 time period) impact of the intervention.

(2) No, there is no permanent impact, because the second part is just a temporary change.

11. Consider an MA(1) model \( Z_t = a_t - \theta a_{t-1} \), where \( \{a_t\} \) is a Gaussian white noise with mean zero and variance \( \sigma_a^2 \). Write down the conditional log likelihood function for the data \( \{Z_1, Z_2, \ldots, Z_T\} \).

A: The log likelihood function is

\[
\ell \propto -\frac{T}{2} \ln(\sigma_a^2) - \frac{1}{2\sigma_a^2} \sum_{t=1}^{T} a_t^2,
\]

where \( a_0 = 0 \) and \( a_t = Z_t + \theta a_{t-1} \).
12. Describe two methods to check a fitted time series model.
   A: Any two of (a) residual plot, (b) Ljung-Box statistics, (c) outlier detection.

13. Suppose that \( Z_t \) is a stationary AR(1) process \( Z_t = \phi Z_{t-1} + a_t \). Suppose also that \( Z_{100} \) and \( Z_{101} \) are two consecutive missing values. What are the least squares estimates \( Z_{100} \) and \( Z_{101} \) based on the sample \( \{ Z_1, Z_2, \ldots, Z_{150} \} \) and the model.
   A: The model gives three equations as below
   \[
   \begin{align*}
   Z_{100} &= \phi Z_{99} + a_{100} \\
   Z_{101} &= \phi Z_{100} + a_{101} \\
   Z_{102} &= \phi Z_{101} + a_{102}.
   \end{align*}
   \]
   From which, we obtain a multiple linear regression as
   \[
   \begin{bmatrix}
   \phi Z_{99} \\
   0 \\
   Z_{102}
   \end{bmatrix}
   =
   \begin{bmatrix}
   1 & 0 \\
   \phi & -1 \\
   0 & \phi
   \end{bmatrix}
   \begin{bmatrix}
   Z_{100} \\
   Z_{101}
   \end{bmatrix}
   +
   \begin{bmatrix}
   -a_{100} \\
   a_{101} \\
   a_{102}
   \end{bmatrix}.
   \]
   Therefore, the least squares estimates of \( Z_{100} \) and \( Z_{101} \) are
   \[
   \begin{bmatrix}
   \hat{Z}_{100} \\
   \hat{Z}_{101}
   \end{bmatrix}
   =
   \begin{bmatrix}
   1 + \phi^2 & -\phi \\
   -\phi & 1 + \phi^2
   \end{bmatrix}^{-1}
   \begin{bmatrix}
   \phi Z_{99} \\
   \phi Z_{102}
   \end{bmatrix}.
   \]

14. Consider an ARIMA model \( \pi(B)Z_t = a_t \), where \( \pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \ldots \). Suppose that there are two consecutive additive outliers in the observed series \( Y_t = \omega_1 I_t^{(100)} + \omega_2 I_t^{(101)} + Z_t \).
   Describe an estimate of the parameter vector \( (\omega_1, \omega_2)' \) that is unbiased.
   A: Let \( e_t = \pi(B)Y_t, x_{1t} = \pi(B)I_t^{(100)} \), and \( x_{2t} = \pi(B)I_t^{(101)} \). In addition, let \( \omega = (\omega_1, \omega_2)' \) and \( x_t = (x_{1t}, x_{2t})' \). Then, using multiple linear regression, we have
   \[
   \hat{\omega} = \left( \sum_{t=100}^{T} x_t x_t' \right)^{-1} \left( \sum_{t=100}^{T} x_t e_t \right).
   \]

15. Define the lag-2 first extended autocorrelation function (EACF) for a time series \( Z_t \).
   A: Lag-2 ACF of \( W_t = Z_t - \frac{\rho_2}{\rho_1} Z_{t-1} \), where \( \rho_i \) is the lag-\( i \) ACF of \( Z_t \).
Problem B. (10 pts) Consider a state-space model

\[
S_{t+1} = FS_t + Ge_t
\]
\[
Z_t = HS_t + \epsilon_t
\]

where \{e_t\} and \{\epsilon_t\} are serially uncorrelated with mean zero, and Cov(e_t) = Q, Cov(\epsilon_t) = R, and Cov(e_t, \epsilon_t) = \Omega. Derive a Kalman filter algorithm for the model.

A: From the model, we obtain

\[
S_{t+1|t} = FS_t|t
\]
\[
P_{t+1|t} = FP_t|tF'| + GQG'
\]
\[
Z_{t+1|t} = HS_{t+1|t}
\]
\[
V_{t+1|t} = HP_{t+1|t}H' + R
\]
\[
C_{t+1|t} = HP_{t+1|t} + \Omega G', \quad C_{t+1|t} = \text{Cov}(Z_{t+1}, S_{t+1}|F_t).
\]

Using the normality of \((S_{t+1}, Z_{t+1})\)' given \(F_t\), we obtain

\[
S_{t+1|t+1} = S_{t+1|t} + C_{t+1|t} V_{t+1|t}^{-1} (Z_{t+1} - Z_{t+1|t})
\]
\[
P_{t+1|t+1} = P_{t+1|t} - C_{t+1|t} V_{t+1|t}^{-1} C_{t+1|t}.
\]

Consequently, a version of Kalman filter is as below: Given \(S_{1|0}\) and \(P_{1|0}\),

\[
Z_{t+1|t} = HS_{t+1|t}
\]
\[
r_{t+1} = Z_{t+1} - Z_{t+1|t}
\]
\[
V_{t+1|t} = HP_{t+1|t}H' + R
\]
\[
C_{t+1|t} = HP_{t+1|t} + \Omega G'
\]
\[
S_{t+2|t+1} = FS_{t+1|t+1} = FS_{t+1|t} + FC_{t+1|t} V_{t+1|t}^{-1} r_{t+1}
\]
\[
P_{t+2|t+1} = FP_{t+1|t+1}F' + GQG'.
\]
Problem C. (24 pts) This problem is concerned with analysis of the monthly U.S. producer price index (PPI): finished goods from January 1977 to October 2005. The data obtained from the Federal Reserve Bank at St. Louis are seasonally adjusted and with Index 1982 = 100. We took the logarithm to stabilize the variability. The output of Splus and SCA is attached. All tests are based on the 5% significance level. Answer the following questions:

1. Is there a unit root in the log series of PPI? Setup the null and alternative hypotheses and perform the test. Draw the conclusion.
   A: Let $Z_t = \ln(PPI_t)$. The model fitted is
   \[ Z_t = \phi Z_{t-1} + \sum_{i=1}^{8} \phi_i \Delta Z_{t-i} + \epsilon_t, \]
   where $\Delta Z_t = Z_t - Z_{t-1}$. The hypotheses are $H_0: \phi = 1$ versus $H_a: \phi < 1$. The test statistic is the t-ratio of $\phi - 1$, which is $-2.324$ with p-value 0.165. Thus, cannot reject the unit-root hypothesis.

2. Is there a double unit root in the log PPI series? Why?
   A: No, the unit-root test rejects the null for the differenced data.

3. Write down the fitted model without outlier detection.
   \[ (1 - .32B - .102B^3 - .123B^5 - .193B^7)(1 - B)Z_t = .0003 + a_t. \]

4. (2 pts) How many outliers are detected based on the prior model?
   A: 13 outliers.

5. (6 pts) Write down the final model with outlier detection. How do you use the model to produce forecasts?
   \[ Y_t = \sum_{i=1}^{13} \omega_i V_i(B) I_t^{(h_i)} + Z_t, \]
   where $v_i(B)$ depends on the type of outlier identified, and $Z_t$ follows the model
   \[ (1 - .322B - .172B^3 - .153B^5 - .191B^7)(1 - B)Z_t = 0.003 + a_t, \]
   where $\sigma_a = .00137$.
   To compute forecasts, do the following:
   - Compute $\omega_i v_i(B) I_t^{(h_i)}$ for $t = T + \ell$, where $\ell > 0$.
   - Compute the forecasts of $Z_{T+\ell}$.
   Combine the two values to produce forecasts.

6. Is the final model adequate? Why?
   A: Yes, based on the Ljung-Box statistics $Q(24) = 29.2$, which is insignificant.
Problem D. (6 pts) Consider the model
\[ Z_t = Z_{t-1} + u_t, \quad u_t = (1 - \theta_1 B - \theta_2 B^2) a_t, \]
where \( Z_0 = 0 \), and \( \{ a_t \} \) is a sequence of independent and identically distributed random variables with mean zero and variance 1. Suppose that we fit an AR(1) model
\[ Z_t = \phi Z_{t-1} + e_t, \]
and let
\[ \hat{\phi} = \frac{\sum_{t=1}^{T} Z_t Z_{t-1}}{\sum_{t=1}^{T} Z_{t-1}^2} \]
be the ordinary least squares estimate of \( \phi \), where \( T \) is the sample size. A test statistic for testing the null hypothesis \( H_0 : \phi = 1 \) is \( T(\hat{\phi} - 1) \). What is the limiting distribution of the test statistic as the sample size \( T \to \infty \).

A: The keys are to obtain \( \sigma^2 = \lim_{T \to \infty} T^{-1} E(S_t^2) \) and \( \sigma_u^2 = p \lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} u_t^2 \).

Because \( u_t = (1 - \theta_1 B - \theta_2 B^2) a_t \), with \( \text{Var}(a_t) = 1 \), we obtain (a) \( \sigma_u^2 = \text{Var}(u_t) = 1 + \theta_1^2 + \theta_2^2 \), and (b) \( \sigma^2 = 1 + \theta_1^2 + \theta_2^2 - 2\theta_1 - 2\theta_2 + 2\theta_1\theta_2 \).

Therefore,
\[ T(\hat{\phi} - 1) \to_D \frac{1}{2} \left[ W(1)^2 - \frac{\sigma_u^2}{\sigma^2} \right] \left[ \int_0^1 W(r)^2 dr \right]^{-1}, \]
where \( \sigma_u^2 \) and \( \sigma^2 \) are given above.