The solution is brief.

**Problem A:** (72 pts) Let \( \{a_t\} \) be a sequence of independent and identically distributed random variables with mean zero and variance \( \sigma_a^2 \), \( \{Z_t\} \) be a time series, and \( B \) the back-shift operator such that \( BZ_t = Z_{t-1} \). Also all forecasts are based on the minimum mean squared error criterion. Briefly answer the following questions.

1. What are the conditions that the series \((1 - \phi_1 B - \phi_2 B^2)Z_t = \phi_0 + a_t\) is weakly stationary?
   
   Answer: All zeros of \( \Phi(B) \) lie outside the unit circle, where \( \Phi(B) \equiv (1 - \phi_1 B - \phi_2 B^2) \).

2. What are the mean and variance of the process \((1 - 0.75B)Z_t = 0.25 + a_t\), if \( \sigma_a^2 = 1.0\)?
   
   Answer: \( \text{mean}=0.25/(1-0.75)=1; \text{Variance}=\sigma_a^2/(1 - 0.75^2) = 2.29 \)

3. Assume that \( \sigma_a^2 = 3.0 \). What are the variance and covariance of the process \( Z_t = 1.0 + a_t - 0.4a_{t-1} \)?
   
   Answer: \( \gamma_0 = (1 + 0.4^2) \ast 3 = 3.48, \gamma_1 = -0.4 \ast 3 = -1.2, \) and \( \gamma_t = 0 \) if \( t > 1 \).

4. Assume that \( (1 - 1.3B + 0.4B^2)Z_t = a_t \), where the variance of \( a_t \) is 1.0. What are the first two lags of the autocorrelation function of \( Z_t \)?
   
   Answer: (lecture 3, page 3). \( \rho_0 = 1, \rho_1 = 1/(1 + 0.4) = 0.9286, \rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0 = 0.807 \).

5. Suppose that \((1 - B)Z_t = 1 + a_t\), where \( \sigma_a^2 = 2.0 \), and at the forecast origin \( T = 100 \), we have 1-step ahead forecast \( Z_{100}(1) = 20 \). Compute the 2-step and 3-step ahead forecasts \( Z_{100}(2) \) and \( Z_{100}(3) \).
   
   Answer: \( Z_t = Z_{t-1}+1+a_t \), therefore, \( Z_{100}(2) = Z_{100}(1)+1 = 21; Z_{100}(3) = Z_{100}(2)+1 = 22 \).

6. Suppose that \((1 - \phi B)Z_t = (1 - \theta B)a_t\), where \( \phi \neq \theta \), and \( Z_t \) is stationary and invertible. What is the model of \( y_t \), where \( y_t = Z_{2t} + Z_{2t-1} \)? It suffices to write down the order.
   
   Answer: an ARMA(1,1) model. [More specifically, it is in the form \((1-\phi^2 B)y_t = (1-\theta B)b_t\).

7. Suppose \( Z_t = X_t + Y_t \), where \( X_t = 0.8X_{t-1} + a_t \) and \( Y_t = 0.5Y_{t-1} + b_t \) with \( \{b_t\} \) being a white noise series independent of \( \{a_t\} \) and \( \text{Var}(b_t) = \sigma_b^2 \). What is the model of \( Z_t \)? It suffices to write down the order of the model.
   
   Answer: an ARMA(2,1) model. [More specifically,

   \[
   (1 - 0.8B)(1 - 0.5B)Z_t = (1 - 0.5B)a_t + (1 - 0.8B)b_t = a_t - 0.5a_{t-1} + b_t - 0.8b_{t-1},
   \]
8. Suppose that \( Z_t = 1.0 + (1 - 0.87B + 0.27B^2)a_t \) and \( a_{99} = 0.3 \) and \( a_{100} = -0.5 \). Also, \( a_t \)

is Gaussian with \( \sigma_a^2 = 1.0 \). Compute the 1-step ahead 95% interval forecast of \( Z_{101} \) at the

forecast origin \( T = 100 \).

Answer: \( Z_{100}(1) = 1.0 - 0.87a_{100} + 0.27a_{99} = 1.516 \); \( \text{Var}(e_{100}(1)) = \sigma_a^2 = 1.0 \). 95% interval

forecast = (1.516-1.96, 1.516+1.96) = (-0.444, 3.476).

9. Suppose that \( (1 - \phi B)Z_t = (1 - \theta B)a_t \) is a stationary and invertible series. At the forecast

origin \( T \), what is the 2-step ahead forecast error? What is the variance of the 2-step ahead

forecast error?

Answer: \( e_T(2) = \sum_{i=0}^{1} \psi_i a_{T+i} = a_{T+2} + (\phi - \theta)a_{T+1} \). \( \text{Var}(e_T(2)) = [1 + (\phi - \theta)^2] \sigma_a^2 \).

10. Obtain all non-zero ACF of the model \( Z_t = (1 - 0.5B - 0.8B^2)a_t \), where \( \text{Var}(a_t) = 2.0 \).

Answer: \( \rho_1 = -0.5/(1 + 0.5^2 + 0.8^2) = -0.2646 \), \( \rho_{11} = 0.5 \times 0.8/(1 + 0.5^2 + 0.8^2) = 0.212 \), and

\( \rho_{12} = -0.8/(1 + 0.5^2 + 0.8^2) = -0.423 \).

11. Consider the difference equation \( (1 - 1.3B + 0.4B^2)Z_t = 0 \). If \( Z_0 = 2.0 \) and \( Z_1 = 1.3 \), what is

the value of \( Z_{15} \)?

Answer: \( Z_t = c_1 \times 0.5^t + c_2 \times 0.8^t \) (lecture 1, page 3). From the initial conditions, \( c_1 + c_2 = 2.0 \)

and \( 0.5c_1 + 0.8c_2 = 1.3 \) so that \( c_1 = c_2 = 1 \). Consequently, \( Z_{15} = 0.0352 \).

12. Suppose that \( (1 - 0.8B)Z_t = a_t \). Let \( Y_t = Z_t - 0.6Z_{t-1} \). What is the model of \( Y_t \)?

Answer: ARMA(1,1) because \( (1 - 0.8B)Y_t = (1 - 0.8B)Z_t - 0.6(1 - 0.8B)Z_{t-1} = a_t - 0.6a_{t-1} \).

Alternatively, \( Z_t = \frac{1}{1 - 0.8B}a_t \) so that \( Y_t = (1 - 0.6B)Z_t = \frac{1 - 0.6B}{1 - 0.8B}a_t \).

13. What is the moment generating function of the model \( (1 - 0.5B)Z_t = (1 - 2B)a_t \), where

\( \text{Var}(a_t) = 1.0 \)?

Answer: \( \Gamma(z) = \frac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})} \sigma^2 = \frac{(1 - 2z)(1 - 2z^{-1})}{(1 - 0.5z)(1 - 0.5z^{-1})} \).

14. Consider a time series \( Z_t \). What is the condition of strong stationarity for \( Z_t \)?

Answer: The joint distribution of \( (Z_t, \ldots, Z_{t+s}) \) is equal to that of \( (Z_{t+r}, \ldots, Z_{t+r+s}) \), \( \forall r, s \). 15. For any linear time series, let \( e_T(\ell) \) be the \( \ell \)-step ahead forecast error at the origin \( T \),

assuming that the model is known. True or false that \( \text{Var}[e_T(\ell)] \geq \text{Var}[e_T(\ell - 1)] \) for \( \ell \geq 2 \)?

Answer: True.

16. Define the AIC criterion for a Gaussian ARMA\((p,q)\) model with sample size \( n \).

Answer: \( \text{AIC}(p,q) = n \log(\hat{\sigma}^2_n) + 2(p+q) \), where \( \hat{\sigma}^2_n \) is the MLE of the variance of the innovation noise.
17. Define the lag-1 first extended autocorrelation function (EACF) for a time series $Z_t$.

Answer: The lag-1 first-order EACF of $Z_t$ is the lag-1 ACF of $W_t = Z_t - \rho_1 Z_{t-1}$, where $\rho_1$ is the lag-$1$ ACF of $Z_t$.

18. Suppose that $Z_t = Z_{t-1} + a_t$. Define $Y_t = Z_{3\ell} + Z_{3\ell-1} + Z_{3\ell-2}$ for $\ell = 1, 2, \ldots$. What is the model for $Y_t$?

Answer: ARMA(1,1) in time scale $\ell$.

Problem B. (16 pts) Consider a time series of 296 observations. Use the computer output (both SCA and R) to answer the following questions:

1. Explain why an AR(3) model is used?

Answer: PACF cuts off after lag 3.

2. Write down the fitted AR(3) model, including the residual standard error.

Answer: $(1 - 1.976B + 1.374B^2 - 0.343B^3)Z_t = a_t$, where $a_t$ is a white noise with mean zero and variance 0.18872.

3. Is the fitted AR(3) model adequate? Why?

Answer: Yes, the residuals have essentially no significant serial correlations. [There is a minor serial correlation at lag-4, but it is small.]

4. What are the 2-step ahead forecast and its standard error at the forecast origin $T=296$?

Answer: $Z_{296}(2) = -0.2193$; Standard error = 0.4179.

Problem C. (12 pts) Consider the monthly series of demand in electricity of a manufacturing sector in the U.S. The series has 264 observations and we analyze the log series. SCA and R outputs are attached. Answer the following questions:
1. Write down the fitted model, including residual standard error.
   Answer: \((1 - B)(1 - B^{12})Z_t = (1 - 0.4869B)(1 - 0.9751B^{12})a_t\), and \(\sigma_a = 0.0182\).

2. Why is the Airline model specified for the series?
   Answer: The sample ACF follows essentially the pattern of an Airline model with non-zero ACF at lags 1, 11, 12 and 13.

3. Discuss briefly the implication of the fitted model. Answer: The AR part of the fitted model consists of the regular and seasonal difference. It says the demand of electricity also follows a seasonal (or periodic) pattern. [In addition, the near cancellation between seasonal AR and seasonal MA factors indicates the seasonal pattern is close being deterministic.]