GSB Honor Code:  
*I pledge my honor that I have not violated the Honor Code during this examination.*

Signature:  
Name:  
UC. ID:

Notes:
- Open notes and books.
- Write your answer in the blank space provided for each question.
- Manage your time carefully and answer as many questions as you can.
- Unless stated otherwise, \( \{a_t\} \) is a sequence of iid Gaussian random variables with mean zero and positive-definite covariance matrix \( \Sigma \). This assumption applies to the univariate case too.
- For simplicity, ALL tests use the 5% significance level.

Problem A: (30 points; 2 points per question) Answer briefly the following questions.

1. **(For problems 1 to 5).** Consider the following 2-dimensional linear time series \( z_t \):

\[
z_t = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i},
\]

where \( \psi_0 = I \), the 2 \( \times \) 2 identity matrix. What is the necessary and sufficient condition that \( z_t \) is weakly stationary?

2. Assume that \( z_t \) is stationary, derive the lag-\( \ell \) autocovariance matrix of \( z_t \) with \( \ell \geq 0 \).

3. Suppose that \( z_t = (z_{1t}, z_{2t})' \). Give a sufficient condition that \( z_{1t} \) does not depend on any past values of \( z_{2t} \), but \( z_{2t} \) depends on some past values of \( z_{1t} \).
4. Given the model, what is the 3-step ahead forecast error of $z_t$ at the forecast origin $t = n$?

5. Given the model, what is the covariance matrix of 3-step ahead forecast errors of the series?

6. (For problems 6 to 7). Suppose that the $k$-dimensional process $z_t$ follows the VARMA(2,1) model

$$(I - \phi_1 B - \phi_2 B^2)z_t = c + (I - \theta_1 B)a_t,$$

where $c$ is a constant vector. What is the necessary and sufficient condition that $z_t$ is weakly stationary?

7. Assuming weak stationarity, write down the moment equations of the $z_t$ series.

8. Consider the transfer function model $y_t = 1.0 + \frac{2.0B^2 + 3.0B^3 - 0.5B^4}{1 - 0.5B} x_t + \frac{1}{1 - 0.6B} a_t$, where $x_t$ is the input variable satisfying $(1 - 0.8B)x_t = (1 + 0.4B)b_t$, and $\{a_t\}$ and $\{b_t\}$ are two independent white noise series with mean zero and variance $\sigma^2_a$ and $\sigma^2_b$, respectively. Derive the first 4 lags of the impulse response function of the model.

9. Consider the unit-root time series $(1 - B)(1 - \phi B)z_t = a_t$, where $|\phi| < 1$. Let $\hat{\rho}_\ell$ be the lag-$\ell$ sample autocorrelation of $z_t$ based on $T$ data points. What is the limiting value of $\hat{\rho}_\ell$ as $T \to \infty$, where $\ell$ is a fixed integer. Briefly outline a proof to justify your statement.
10. Consider the seasonal time series
\[ z_t = \mu + (I - \theta B)(I - \Theta B^{12})a_t. \]
Derive all non-zero autocovariance matrices \( \Gamma - \ell \) of \( z_t \) for \( \ell > 0 \).

11. Define a scalar component model of order (2,1) for the vector time series \( z_t \).

12. (For Problems 12 to 15). Suppose that \( z_t \) follows the model
\[ z_t = \begin{bmatrix} 1.0 & 0.5 \\ -0.3 & 1.8 \end{bmatrix} z_{t-1} - \begin{bmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{bmatrix} z_{t-2} + a_t. \]
Write down the AR polynomial matrix of the model and find the matrix \( \phi(1) \).

13. Show that \( z_{1t} \) and \( z_{2t} \) are unit-root nonstationary.

14. Write down an Error-Correction Model for the system.

15. Write down the co-integrating vector for the system in which the first element is 1.
Problem B. (20 points) Consider a 2-dimensional time series $z_t = (z_{1t}, z_{2t})'$ with 200 observations. The SCA analysis of the data is attached in which $z_{1t}$ and $z_{2t}$ are called “x” and “y”, respectively. Use the output and answer the following questions.

1. (4 points) Identify the order of a VARMA model for the series using either AIC or the Chi-square test statistics.

2. (7 points) A VARMA(1,1) model was entertained because the residual CCM of the AR(1) fit indicates simple serial dependence. In addition, insignificant parameters are removed. Write down the final fitted model, including the residual covariance matrix.

3. (4 points) Based on the fitted model, what is the marginal model for $z_{1t}$?

4. (5 points) Based on the fitted model, discuss the relationship between the two series and write down the transfer function model, if it exists.
Problem C. (10 points) Consider the quarterly U.S. exports and imports of Goods, Services and Income from 1960 to 2006 for 188 observations. The data are in billions of dollars and from the Federal Reserve Banks of St Louis. In addition, the data are seasonally adjusted. We employ the log series of the two time series. S-Plus output is attached.

1. (6 points) Are the two individual log series unit-root nonstationary? Why?

2. (4 points) Are the two time series co-integrated? If yes, what is the number of co-integrating vectors? If not, why?
Problem D. (20 points) Suppose that $z_t$ is a linear stationary process with Kronecker indices $\{k_1 = 2, k_2 = 1, k_3 = 1\}$. Write down the implied VARMA model for $z_t$. How many parameters in the AR and MA polynomials that require estimation?
Problem E. (20 points total, 10 points each question) Simple proofs

1. Consider a $k$-dimensional stationary time series $z_t$. Let $\hat{\Gamma}_\ell$ and $\hat{\rho}_\ell$ be the lag-$\ell$ sample cross covariance and correlation matrix, respectively. The multivariate Ljung-Box statistics for the null hypothesis $H_o : \rho_1 = \cdots = \rho_m = 0$ versus $H_a : \rho_i \neq 0$ for some $i \in \{1, \ldots, m\}$ is defined as

$$Q_k(m) = T^2 \sum_{\ell=1}^{m} \frac{1}{T-\ell} tr(\hat{\Gamma}_\ell \hat{\Gamma}_0^{-1} \hat{\Gamma}_\ell \hat{\Gamma}_0^{-1}),$$

where $T$ is the sample size and $tr$ denotes the trace of a square matrix. Show that $Q_k(m)$ can be written as

$$Q_k(m) = T^2 \sum_{\ell=1}^{m} \frac{1}{T-\ell} b'_\ell (\hat{\rho}_0^{-1} \otimes \hat{\rho}_0^{-1}) b_\ell,$$

where $\otimes$ denotes the Kronecker product and $b_\ell = \text{vec}(\hat{\rho}_\ell')$ with vec being the column-stacking operator.
2. Consider the univariate process

\[ z_t = z_{t-1} + a_t - \theta a_{t-2}, \]

where \( z_0 = 0 \) and \(|\theta| < 1\). Suppose that the sample \( \{z_1, \ldots, z_T\} \) is available. Derive the limiting distribution of \( T^{-2} \sum_{t=1}^{T} z_{t-1}^2 \) as \( T \to \infty \).