Problem A: (45 points; 3 points per question) Answer briefly the following questions.

1. Solutions of $|\theta(B)| = 0$ are all outside the unit circle, i.e., their absolute values are greater than 1.

2. Without loss of generality, we may assume $\phi_0 = 0$. Multiplying the model by $z_{t-\ell}$ with $\ell > q$ and taking expectation, we have

$$\Gamma_\ell = \sum_{i=1}^p \phi_i \Gamma_{\ell-i}, \quad \ell > q.$$  

3. Using the MA representation $z_t = (I + \psi_1 B + \psi_2 B^2 + \cdots) a_t$, where the $\psi$-weight matrices can be obtained by $I + \sum_{i=1}^\infty \psi_i B^i = [\phi(B)]^{-1}\theta(B)$. The 3-step ahead forecast error is $e_{n}(3) = a_{n+3} + \psi_1 a_{n+2} + \psi_2 a_{n+1}$.

4. The covariance matrix of $e_{n}(3) = \Sigma + \psi_1 \Sigma \psi_1' + \psi_2 \Sigma \psi_2'$.

5. There are several possibilities. Any of the followings is fine. (i) $\phi_1$, $\phi_1$ and $\theta_1$ are simultaneously all lower triangular matrices. (ii) They are simultaneously all upper triangular matrices. You can also write $\phi_{ij}(B)$ and $\theta_{ij}(B)$ as the $(i,j)$th element of $\phi(B)$ and $\theta(B)$, respectively. Then, the two additional possibilities are (iii)

$$\begin{vmatrix} \phi_{12}(B) & \theta_{12}(B) \\ \phi_{22}(B) & \theta_{22}(B) \end{vmatrix} = 0,$$

and (iv)

$$\begin{vmatrix} \phi_{12}(B) & \theta_{12}(B) \\ \phi_{22}(B) & \theta_{22}(B) \end{vmatrix} \neq 0.$$

6. Using $\pi(B) = I - \pi_1 B - \pi_2 B^2 - \cdots = (I - \theta B)^{-1}(I - \phi_1 B - \phi_2 B^2)$, we obtain $\pi_1 = \phi_1 - \theta_1$, $\pi_2 = \phi_2 + \theta_1 \pi_1$, and $\pi_j = \theta_1^{j-2}\pi_2$ for $j > 2$.

7. Univariate ARMA(4,3) model.

8. Direct calculation shows that (i) $\Gamma_0 = \Sigma + \theta_1 \Sigma \theta_1'$, $\theta_3 \Sigma \theta_3'$, (ii) $\Gamma_1 = -\theta_1 \Sigma a$, (iii) $\Gamma_2 = \theta_3 \Sigma a \theta_1'$, and $\Gamma_3 = -\theta_3 \Sigma a$.

9. The procedure is as follows: Let $z_t = (x_t, y_t)'$ and assume that we are interested in testing $x_t$ causes $y_t$.

(a) Select a VAR order for $z_t$.

(b) Fit a model for $y_t$ using past values of $x_t$ and $y_t$ and the future values $x_t$. 

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(c) Test that the some coefficients of the past values of $x_t$ are nonzero, but all coefficients of the future values of $x_t$ are zero.

10. From R, the Choleski decomposition of $\Sigma_a$ is $\Sigma_a = M'M$, where $M = \begin{bmatrix} 1 & 0.5 \\ 0 & 1.323 \end{bmatrix}$ so that $(M')^{-1} = \begin{bmatrix} 0.3 & 0 \\ -0.378 & 0.756 \end{bmatrix}$. Pre-multiplying $(M')^{-1}$ to the model, we have

$$(M')^{-1}z_t = \begin{bmatrix} 0.3 & 0 \\ -0.567 & 0.680 \end{bmatrix}a_t = b_t, \quad b_t = M'a_t.$$

Consequently, we have

$$-0.378z_{1t} + 0.756z_{2t} = -0.567z_{1,t-1} + 0.68z_{2,t-1} + b_{2t}$$

or equivalently,

$$-0.5z_{1t} + z_{2t} = -0.75z_{1,t-1} + 0.9z_{2,t-1} + 1.323b_{2t}.$$

Note that one can write $1.323b_{2t}$ as $e_t$ with standard error 1.323.

11. From $\Sigma_a$, $a_{2t} = 0.5a_{1t} + e_t$. Therefore, we have

$$(1-0.9B)z_{2t} = (0.5 - 0.75B)z_{1t} + e_t \quad \text{or} \quad z_{2t} = \frac{0.5 - 0.75B}{1 - 0.9B}z_{1t} + \frac{1}{1 - 0.9B}e_t,$$

where $e_t$ is uncorrelated with $z_{1t}$.

12. From the TF model, we easily obtain $v_0 = 0.5$, $v_1 = -0.3$ and $v_2 = -0.27$, and $v_3 = -0.234$.

13. Since $|\phi(1)| = 0$, the system contains unit root.

14. An error correction form is

$$\Delta z_t = -\begin{bmatrix} 0.54 & 1.08 \\ 1.08 & 2.16 \end{bmatrix}z_{t-1} + \begin{bmatrix} -0.48 & 0.04 \\ 0.04 & -0.42 \end{bmatrix}\Delta z_{t-1} + a_t.$$

15. From the error-correction form, $(1, 2)'$ is a co-integrating vector.

Problem B. Consider the monthly unemployment rates of Illinois and Indiana from January 1976 to March 2009.

1. Based on either AIC or the sequential chi-square test.

2. The fitted model is

$$z_t = \begin{bmatrix} 0.128 \\ 0.138 \end{bmatrix} + \begin{bmatrix} 1.107 & .132 \\ 0 & 1.202 \end{bmatrix}z_{t-1} + \begin{bmatrix} 0 & -.086 \\ .202 & -.183 \end{bmatrix}z_{t-2} + \begin{bmatrix} 0 \\ -0.237 \end{bmatrix}z_{t-3} + \begin{bmatrix} -0.165 \\ 0 \end{bmatrix}z_{t-4} + a_t,$$

where $\text{cov}(a_t) = \begin{bmatrix} .028 & .012 \\ .012 & .058 \end{bmatrix}$.
3. The two unemployment rate series show inter-dependence between them, that is, there is feedback relationship between them.

4. These are 4-step ahead predictions. The prediction intervals are $10.354 \pm 1.96 \times 0.430$ and $10.719 \pm 1.96 \times 0.594$, respectively.

**Problem C.** Again, consider the monthly unemployment rates of Illinois and Indiana from January 1976 to March 2009.

1. Yes, the augmented Dickey-Fuller tests fail to reject the null hypothesis of unit-root time series.

2. Yes, the co-integration test shows that the two unemployment rate series are co-integrated. A co-integrating vector is $(-1.414, 1.119)'$.

**Problem D.** (10 points) Consider a univariate AR($p$) model $$(1 - \phi_1 B - \cdots - \phi_p B^p) z_t = a_t.$$  

1. Solutions of $\phi(B) = 0$ are all greater than one in modulus.

2. Define $Z_t = (z_t, z_{t-1}, \ldots, z_{t-p+1})'$ and $b_t = (a_t, 0, \ldots, 0)'$. It is easy to see that

$$Z_t = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} Z_{t-1} + b_t.$$  

3. Note that $z_t$ is stationary if and only if $Z_t$ is stationary. For VAR(1) model, $Z_t$ is stationary if all eigenvalues of $\Phi$ are less than one in modulus.

4. If $z_t$ is unit-root nonstationary, then 1 is a solution to the polynomial $\phi(B) = 0$. That is, $\phi(1) = 0$. Consequently, $\sum_{t=1}^{p} \phi_t = 1$.

**Problem E.** (20 points total, 5 points each question) Simple derivation.

1. **Proof** of (1a). Let $\sigma^2 = \lim_{T \to \infty} E(T^{-1} S_T^2)$. Since $a_t$ is a white noise, we have $\sigma^2 = \sigma_a^2$.

$$T^{-3/2} \sum_{t=1}^{T} z_t = T^{-3/2} \sum_{t=1}^{T} (S_{t-1} + a_t + z_0) = T^{-3/2} \sum_{t=1}^{T} S_{t-1} + T^{-1/2} [\frac{1}{T} \sum_{t=1}^{T} a_t] + T^{-1/2} z_0$$

$$= \sigma \sum_{t=1}^{T} \frac{1}{\sqrt{T}} S_t \frac{1}{T} + T^{-1/2} \bar{a} + T^{-1/2} z_0$$

$$= \sigma \sum_{t=1}^{T} \int_{(t-1)/T}^{t/T} \frac{1}{\sqrt{T}} S[T] dr + T^{-1/2} \bar{a} + T^{-1/2} z_0$$

$$= \sigma \int_{0}^{1} X_T(r) dr + T^{-1/2} \bar{a} + T^{-1/2} z_0$$

$$\Rightarrow \sigma \int_{0}^{1} W(r) dr.$$
Proof of (1b). From \( z_t = z_{t-2} + a_{t-1} + a_t \), \( z_t^2 = z_{t-2}^2 + a_{t-1}^2 + a_t^2 + 2a_{t-1}a_t + 2z_{t-2}a_{t-1} + 2z_{t-2}a_t \). Therefore, \( 2z_{t-2}a_t = z_t^2 - z_{t-2}^2 - a_{t-1}^2 - a_t^2 - 2z_{t-2}a_{t-1} - 2a_{t-1}a_t \). Summing over \( t \), we have

\[
\sum_{t=2}^{T} z_{t-2}a_t = \frac{1}{2}[z_T^2 + z_{T-1}^2 - z_1^2 - z_0^2 - \sum_{t=2}^{T} a_{t-1}^2 - \sum_{t=2}^{T} a_t^2] - \sum_{t=2}^{T} z_{t-2}a_{t-1} - \sum_{t=2}^{T} a_{t-1}a_t.
\]

Consequently,

\[
T^{-1} \sum_{t=2}^{T} z_{t-2}a_t = \frac{1}{2}[T^{-1}z_T^2 + T^{-1}z_{T-1}^2 - T^{-1}(z_1^2 + z_0^2) - T^{-1}\sum_{t=2}^{T} a_{t-1}^2 - T^{-1}\sum_{t=2}^{T} a_t^2]
- T^{-1}\sum_{t=2}^{T} z_{t-2}a_{t-1} - T^{-1}\sum_{t=2}^{T} a_{t-1}a_t.
\]

Applying FCLT and the law of large numbers, we have

\[
T^n \sum_{t=2}^{T} z_{t-2}a_t \Rightarrow \frac{1}{2} \left[ \sigma^2 W(1)^2 + \sigma^2 W(1)^2 - \sigma_a^2 \sigma_a^2 \right] - \frac{\sigma^2}{2} [W(1)^2 - \sigma_x^2],
\]

After simplification,

\[
T^{-1} \sum_{t=2}^{T} z_{t-2}a_t \Rightarrow \frac{\sigma^2}{2} [W(1)^2 - \frac{\sigma_a^2}{\sigma^2}] = \frac{\sigma_a^2}{2} [W(1)^2 - 1],
\]

which is the same as that of \( T^{-1} \sum_{t=1}^{T} z_{t-1}a_t \).

For problem 2, we follow the problem in HW assignment. It involves four independent standard Brownian motions. Write \( z_t = z_{4i+j} \) where \( i = 1, 2, 3, 4 \) and \( j = 0, 1, 2, \ldots \). Also, recall that for the model \( (1 - B) y_t = (1 - \theta B) a_t \), we have

\[
\sum_{t=1}^{T} \frac{y_t^2}{n} \Rightarrow \sigma^2 \int_0^1 W(r)^2 dr, \quad T^{-1} \sum_{t=1}^{T} y_{t-1}a_t \Rightarrow \frac{\sigma_a^2}{2} [W(1)^2 - \frac{\sigma_x^2}{\sigma_a^2}],
\]

where \( \sigma^2 = (1 - \theta)^2 \sigma_a^2 \) and \( \sigma_x^2 = (1 + \theta^2) \sigma_a^2 \), where \( x_t = a_t - \theta a_{t-1} \).

Proof of (2a).

\[
T^{-2} \sum_{t=1}^{T/4-1} z_{t-4}^2 = T^{-2} \sum_{i=0}^{(T/4)-1} \sum_{j=1}^{4} z_{4i+j}^2
\rightarrow \frac{1}{4^2} (1 - \theta)^2 \sigma_a^2 \sum_{j=1}^{4} \int_0^1 W_j^2(r) dr.
\]

Proof of (2b).

\[
T^{-1} \sum_{t=1}^{T} z_{t-4}a_t = T^{-1} \sum_{i=0}^{T/4} \sum_{j=1}^{4} z_{4i+j}a_{4(i+1)+j}
\rightarrow \frac{1}{4} (1 - \theta)^2 \sigma_a^2 \sum_{j=1}^{4} [W_j^2(1) - \frac{1 + \theta^2}{(1 - \theta)^2}].
\]