Solutions to Homework Assignment 1

1. The file “data-1.txt” has two columns. Each column contains a time series with 400 observations.

   - Build a univariate ARIMA model for each time series.
   - Perform a test to identify the input variable, i.e. the exogenous variable. You may use 5 lags of the other (i.e., output) variable to perform the test.
   - Build a transfer function model for the data.
   - Re-estimate the ARIMA model for the output variable and the transfer function model using the first 350 data points. Use the last 50 data points for forecasting comparison. Conduct 50 1-step ahead forecasts in the forecasting subsample. Calculate the root mean squares of forecast errors for both models. Comment on the results of comparison.

   **Answer:** Let $x_t$ and $y_t$ be the first and 2nd column of the data.

   - $x_t$ follows an AR(1) model $(1 - 0.57B)x_t = a_{1t}$ with residual standard error 3.40. $y_t$ follows an AR(2) model $(1 - 1.34B + .45B^2)y_t = -0.04 + a_{2t}$ with residual standard error 0.97.
   - I ran two regression models using (a) own past five lags and (b) past and future 5 lags of the other variable. From the output, it is clear that $x_t$ is the output variable and $y_t$ the input variable.
   - The transfer function model built is
     
     $$x_t = (1.99 - 2.98B)y_t + (1 - 0.60B)a_t, \quad \sigma_a = 1.04.$$  
     
     This model is adequate based on model checking.
   - Focus on the output variable. The out-of-sample forecasts for the last 50 data points show that the mean squared errors of forecasts for the AR(1) and transfer function models are 0.96 and 9.22, respectively. Clearly the transfer function model is preferred.

2. Consider the transfer function model

   $$Y_t = (\beta_0 + \beta_2B^2)X_t + N_t,$$
   
   where $N_t$ follows the model $(1 - B)N_t = (1 - \theta B)a_t$, where $|\theta| < 1$. Suppose further that $X_t$ follows the model $(1 - \phi_1B)X_t = (1 - \omega B)b_t$, where \{b_t\} is independent of \{a_t\} and $\phi_1 \neq \omega$. The two innovation series \{a_t\} and \{b_t\} are white noise with mean zero and unit variance and they are independent. Let $Z_t = (X_t, Y_t)'$. Find a vector ARMA model for $Z_t$.  

**Answer:** From the models, we obtain

\[ Y_t = Y_{t-1} + (\beta_0 - \beta_0 B + \beta_2 B^2 - \beta_2 B^3) X_t + (1 - \theta B) a_t, \]
\[ X_t = \phi_1 X_{t-1} + b_t - \omega b_{t-1}. \]

Consequently, we can write down a VARMA(3,1) model for \( Z_t = (X_t, Y_t)' \). Let

\[ P = \begin{bmatrix} 1 & -\beta_0 \\ 0 & 1 \end{bmatrix}, \phi_1 = \begin{bmatrix} 1 & -\beta_0 \\ 0 & \phi_1 \end{bmatrix}, \]

\( \phi_2 \) and \( \phi_3 \) are zero except for the (1,2)th element. The values of the (1,2)th element are \( \beta_2 \) and \( -\beta_2 \), respectively, and \( \theta = \text{diag}\{\theta, \omega}\). The ARMA coefficient matrices are \( P^{-1} \phi_i \) and \( P^{-1} \theta \).

**SCA and R** output is on the course web.