Multivariate Volatility Models

How do the correlations between asset returns evolve over time?

Focus on two series (Bivariate)
Two asset return series:
\[
\mathbf{r}_t = \begin{bmatrix} r_{1t} \\ r_{2t} \end{bmatrix}.
\]

Data: \(r_1, r_2, \ldots, r_T\).

Basic concept
Let \(F_{t-1}\) denote the information available at time \(t-1\).
Partition the return as
\[
\mathbf{r}_t = \mathbf{\mu}_t + \mathbf{a}_t, \quad \mathbf{a}_t = \Sigma_t^{1/2} \mathbf{\epsilon}_t
\]
where \(\mathbf{\mu}_t = \mathbb{E}(\mathbf{r}_t|F_{t-1})\) is the predictable component, and
\[
\text{Cov}(\mathbf{a}_t|F_{t-1}) = \Sigma_t = \begin{bmatrix} \sigma_{11,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{22,t} \end{bmatrix},
\]
\(\{\mathbf{\epsilon}_t\}\) are iid 2-dimensional random vectors with mean zero and identity covariance matrix.

Multivariate volatility modeling
Study time evolution of \(\{\Sigma_t\}\).

\(\Sigma_t\) is symmetric, i.e. \(\sigma_{12,t} = \sigma_{21,t}\)
There are 3 variables in \(\Sigma_t\).
If \(k\) asset returns, \(\Sigma_t\) has \(k(k+1)/2\) variables.

Requirement
\(\Sigma_t\) must be positive definite for all \(t\),
\[
\sigma_{11,t} > 0, \quad \sigma_{22,t} > 0, \quad \sigma_{11,t}\sigma_{22,t} - \sigma_{12,t}^2 > 0.
\]

The time-varying correlation between \(r_{1t}\) and \(r_{2t}\) is
\[
\rho_{12,t} = \frac{\sigma_{12,t}}{\sqrt{\sigma_{11,t}\sigma_{22,t}}}.
\]
Some complications when the dimension is high

- Positiveness requirement is not easy to meet
- Too many series to model

Some simple models available

- Exponentially weighted covariance
- Diagonal VEC models
- BEKK models
- Dynamic correlation models (a) Tse and Tsui (2002) and (b) Engle (2002)

Exponentially weighted model

\[ \Sigma_t = (1 - \lambda) a_{t-1} a'_{t-1} + \lambda \Sigma_{t-1}, \]

where \( 0 < \lambda < 1 \). That is,

\[ \Sigma_t = (1 - \lambda) \sum_{i=1}^{\infty} \lambda^{i-1} a_{t-i} a'_{t-i}. \]

cov1 = EWMA.cov(rtn, lambda=0.96) % lambda given.
cov2 = mgarch(rtn~1,~ewma1,trace=F) % Estimate lambda

Diagonal VEC model

May not be positive definite.

Model elements of \( \Sigma_t \) separately

For instance, DVEC(1,1) model

\[
\begin{align*}
\sigma_{11,t} &= c_{11} + \alpha_{11} a_{1,t-1}^2 + \beta_{11} \sigma_{11,t-1} \\
\sigma_{12,t} &= c_{12} + \alpha_{12} a_{1,t-1} a_{2,t-1} + \beta_{12} \sigma_{12,t-1} \\
\sigma_{22,t} &= c_{22} + \alpha_{22} a_{2,t-1}^2 + \beta_{22} \sigma_{22,t-1}
\end{align*}
\]

In S-Plus, \( c \) is called “A”, \( \alpha \) is ARCH(1), \( \beta \) is GARCH(1).

fit = mgarch(rtn~1,~dvec(1,1))
summary(fit)
names(fit) % see what are stored.
    %In particular, fit$R.t stores correlations
**BEKK model** Engle and Kroner (1995)
Simple BEKK(1,1) model

\[ \Sigma_t = A_0 A_0' + A_1 (a_{t-1} a_{t-1}') A_1' + B_1 \Sigma_{t-1} B_1' \]

where \( A_0 \) is a lower triangular matrix, \( A_1 \) and \( B_1 \) are square matrices without restrictions.

Pros: positive definite
Cons: Many parameters, dynamic relations require further study

```
fit2=mgarch(rtn~1,"bekk(1,1))
summary(fit2)
names(fit2)
```

Consider the monthly log returns of GM stock and the S&P500 index from 1950 to 2002 for 638 data points.

*** Demonstration of S-Plus package

```R
> gm=gmspln5002$gm
> sp=gmspln5002$sp
> par(mfcol=c(2,1))
> plot(gm,type='l')
> plot(sp,type='l')
> gmsp=cbind(gm,sp) % create a 2-dim return series.
> fit1=mgarch(gmsp~1,"dvec(1,1),trace=F)
> summary(fit1)
```

*Mean Equation: gmsp ~ 1*
*Conditional Variance Equation: ~ dvec(1, 1)*
*Conditional Distribution: gaussian*

----------------------------------------

**Estimated Coefficients:**
----------------------------------------

| Parameter | Value   | Std.Error | t value | Pr(>|t|) |
|-----------|---------|-----------|---------|---------|
| C(1)      | 0.0104749 | 0.00243210 | 4.307   | 1.919e-005 |
| C(2)      | 0.0068800 | 0.00153509 | 4.482   | 8.790e-006 |
| A(1, 1)   | 0.0001400 | 0.00004496 | 3.113   | 1.933e-003 |
| A(2, 1)   | 0.0001073 | 0.00004952 | 2.168   | 3.055e-002 |
| A(2, 2)   | 0.0001033 | 0.00003944 | 3.840   | 9.018e-003 |
| ARCH(1; 1, 1) | 0.0831688 | 0.01826946 | 4.552   | 6.367e-006 |
| ARCH(1; 2, 1) | 0.0385129 | 0.01622698 | 2.373   | 1.792e-002 |
| ARCH(1; 2, 2) | 0.0802303 | 0.02089382 | 3.840   | 1.355e-004 |
| GARCH(1; 1, 1) | 0.8916125 | 0.02274390 | 39.202  | 0.000e+000 |
| GARCH(1; 2, 1) | 0.9029094 | 0.03801639 | 23.751  | 0.000e+000 |
GARCH(1; 2, 2) 0.8656909 0.03383195 25.588 0.000e+000
---------------------------------------------------------------------
AIC(11) = -4269.022
BIC(11) = -4220.015

Normality Test:
---------------------------------------------------------------------
<table>
<thead>
<tr>
<th></th>
<th>Jarque-Bera</th>
<th>P-value</th>
<th>Shapiro-Wilk</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>44.42</td>
<td>2.266e-10</td>
<td>0.9869</td>
<td>0.6761</td>
</tr>
<tr>
<td>sp</td>
<td>21.52</td>
<td>2.122e-05</td>
<td>0.9892</td>
<td>0.8954</td>
</tr>
</tbody>
</table>

Ljung-Box test for standardized residuals:
---------------------------------------------------------------------
<table>
<thead>
<tr>
<th></th>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>19.67</td>
<td>0.07364</td>
<td>12</td>
</tr>
<tr>
<td>sp</td>
<td>19.98</td>
<td>0.06740</td>
<td>12</td>
</tr>
</tbody>
</table>

Ljung-Box test for squared standardized residuals:
---------------------------------------------------------------------
<table>
<thead>
<tr>
<th></th>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>15.652</td>
<td>0.2077</td>
<td>12</td>
</tr>
<tr>
<td>sp</td>
<td>4.999</td>
<td>0.9580</td>
<td>12</td>
</tr>
</tbody>
</table>

Lagrange multiplier test:
---------------------------------------------------------------------
<table>
<thead>
<tr>
<th>Lag 1</th>
<th>Lag 2</th>
<th>Lag 3</th>
<th>Lag 4</th>
<th>Lag 5</th>
<th>Lag 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>-0.06115</td>
<td>-0.2803</td>
<td>-1.4642</td>
<td>-0.4926</td>
<td>1.1335</td>
</tr>
<tr>
<td>sp</td>
<td>-0.02731</td>
<td>-0.4904</td>
<td>-0.5857</td>
<td>-0.9897</td>
<td>0.5748</td>
</tr>
<tr>
<td>Lag 7</td>
<td>Lag 8</td>
<td>Lag 9</td>
<td>Lag 10</td>
<td>Lag 11</td>
<td>Lag 12</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>gm</td>
<td>-0.2745</td>
<td>1.2502</td>
<td>0.9645</td>
<td>-0.8264</td>
<td>2.1744</td>
</tr>
<tr>
<td>sp</td>
<td>-0.1627</td>
<td>0.4246</td>
<td>0.3809</td>
<td>1.4249</td>
<td>0.2165</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TR^2</th>
<th>P-value</th>
<th>F-stat</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>15.035</td>
<td>0.2395</td>
<td>1.4006</td>
</tr>
<tr>
<td>sp</td>
<td>4.739</td>
<td>0.9661</td>
<td>0.4341</td>
</tr>
</tbody>
</table>

> fit2=mgarch(gmsp~1,"bekk(1,1)") % BEKK model
> summary(fit2)
mgarch(formula.mean=gmsp~1,formula.var="bekk(1,1)"

Mean Equation: gmsp ~ 1
Conditional Variance Equation: ~ bekk(1, 1)
Conditional Distribution: gaussian
---------------------------------------------------------------------
Estimated Coefficients:
|                | Value | Std.Error | t value | Pr(>|t|) |
|----------------|-------|-----------|---------|----------|
| C(1)           | 0.0101123 | 0.002697 | 3.74967 | 1.934e-004 |
| C(2)           | 0.0065510 | 0.001697 | 3.86051 | 1.248e-004 |
| A(1, 1)        | 0.0119214 | 0.007089 | 1.68176 | 9.311e-002 |
| A(2, 1)        | 0.0146959 | 0.004269 | 3.44261 | 6.143e-004 |
| A(2, 2)        | 0.0007941 | 0.056368 | 0.01409 | 9.888e-001 |
| ARCH(1; 1, 1)  | 0.3348499 | 0.052971 | 6.32135 | 4.910e-010 |
| ARCH(1; 2, 1)  | 0.0814715 | 0.039972 | 2.03820 | 4.195e-002 |
| ARCH(1; 1, 2)  | -0.3081572 | 0.078010 | -3.95023 | 8.690e-005 |
| ARCH(1; 2, 2)  | 0.1630736 | 0.054573 | 2.98816 | 2.916e-003 |
| GARCH(1; 1, 1) | 0.9237848 | 0.020397 | 45.29098 | 0.000e+000 |
| GARCH(1; 2, 1) | -0.0280710 | 0.016658 | -1.68515 | 9.245e-002 |
| GARCH(1; 1, 2) | 0.0710387 | 0.047735 | 1.48818 | 1.372e-001 |
| GARCH(1; 2, 2) | 0.9304805 | 0.031737 | 29.31835 | 0.000e+000 |

AIC(13) = -4241.851  
BIC(13) = -4183.933  

Normality Test:

<table>
<thead>
<tr>
<th></th>
<th>Jarque-Bera</th>
<th>P-value</th>
<th>Shapiro-Wilk</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>41.87</td>
<td>8.086e-010</td>
<td>0.9901</td>
<td>0.9418</td>
</tr>
<tr>
<td>sp</td>
<td>36.84</td>
<td>1.001e-008</td>
<td>0.9868</td>
<td>0.6660</td>
</tr>
</tbody>
</table>

Ljung-Box test for standardized residuals:

<table>
<thead>
<tr>
<th></th>
<th>Statistic</th>
<th>P-value</th>
<th>Chi(^2)-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>21.68</td>
<td>0.04123</td>
<td>12</td>
</tr>
<tr>
<td>sp</td>
<td>20.98</td>
<td>0.05067</td>
<td>12</td>
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</tbody>
</table>

Ljung-Box test for squared standardized residuals:

<table>
<thead>
<tr>
<th></th>
<th>Statistic</th>
<th>P-value</th>
<th>Chi(^2)-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>20.540</td>
<td>0.05754</td>
<td>12</td>
</tr>
<tr>
<td>sp</td>
<td>7.639</td>
<td>0.81264</td>
<td>12</td>
</tr>
</tbody>
</table>

Lagrange multiplier test:

<table>
<thead>
<tr>
<th></th>
<th>Lag 1</th>
<th>Lag 2</th>
<th>Lag 3</th>
<th>Lag 4</th>
<th>Lag 5</th>
<th>Lag 6</th>
<th>Lag 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>0.09856</td>
<td>-0.1232</td>
<td>-1.1413</td>
<td>-0.4392</td>
<td>1.279</td>
<td>-1.6702</td>
<td>-0.2185</td>
</tr>
<tr>
<td>sp</td>
<td>0.77406</td>
<td>-0.4373</td>
<td>-0.4633</td>
<td>-0.2767</td>
<td>1.181</td>
<td>-0.2194</td>
<td>0.6637</td>
</tr>
</tbody>
</table>

TR\(^2\) P-value F-stat P-value

<table>
<thead>
<tr>
<th></th>
<th>Lag 1</th>
<th>Lag 2</th>
<th>Lag 3</th>
<th>Lag 4</th>
<th>Lag 5</th>
<th>Lag 6</th>
<th>Lag 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>19.903</td>
<td>0.06895</td>
<td>1.8690</td>
<td>0.1201</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sp</td>
<td>6.963</td>
<td>0.86002</td>
<td>0.6402</td>
<td>0.8942</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
> fit3 = mgarch(gmsp ~ 1, ~ccc(1,1)) # Constant correlations!
> summary(fit3)
mgarch(formula.mean = gmsp ~ 1, formula.var = ~ccc(1,1))

Mean Equation: gmsp ~ 1
Conditional Variance Equation: ~ ccc(1, 1)
Conditional Distribution: gaussian

Estimated Coefficients:

|   | Value       | Std.Error | t value | Pr(>|t|) |
|---|-------------|-----------|---------|----------|
| C(1) | 0.01049566 | 0.00241094 | 4.353 | 1.564e-005 |
| C(2) | 0.00735053 | 0.00152773 | 4.811 | 1.876e-006 |
| A(1, 1) | 0.00007338 | 0.00004950 | 1.482 | 1.387e-001 |
| A(2, 2) | 0.00007252 | 0.00003277 | 2.213 | 2.727e-002 |
| ARCH(1; 1, 1) | 0.11404920 | 0.02685448 | 4.247 | 2.493e-005 |
| ARCH(1; 2, 2) | 0.09837054 | 0.02328916 | 4.224 | 2.755e-005 |
| GARCH(1; 1, 1) | 0.88296850 | 0.02786099 | 31.692 | 0.000e+000 |
| GARCH(1; 2, 2) | 0.86944221 | 0.02880646 | 30.182 | 0.000e+000 |

Estimated Conditional Constant Correlation Matrix:

<table>
<thead>
<tr>
<th></th>
<th>gm</th>
<th>sp</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>1.0000</td>
<td>0.6482</td>
</tr>
<tr>
<td>sp</td>
<td>0.6482</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Standard Errors:

<table>
<thead>
<tr>
<th></th>
<th>gm</th>
<th>sp</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>NA</td>
<td>0.01906</td>
</tr>
<tr>
<td>sp</td>
<td>0.01906</td>
<td>NA</td>
</tr>
</tbody>
</table>

AIC(8) = -4238.222
BIC(8) = -4202.58

Normality Test:

<table>
<thead>
<tr>
<th></th>
<th>Jarque-Bera</th>
<th>P-value</th>
<th>Shapiro-Wilk</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>gmsp</td>
<td>35.44</td>
<td>2.015e-008</td>
<td>0.9867</td>
<td>0.6564</td>
</tr>
<tr>
<td>sp</td>
<td>36.80</td>
<td>1.022e-008</td>
<td>0.9852</td>
<td>0.4492</td>
</tr>
</tbody>
</table>

Ljung-Box test for standardized residuals:

<table>
<thead>
<tr>
<th></th>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>gmsp</td>
<td>18.05</td>
<td>0.11415</td>
<td>12</td>
</tr>
<tr>
<td>sp</td>
<td>22.72</td>
<td>0.03016</td>
<td>12</td>
</tr>
</tbody>
</table>
Ljung-Box test for squared standardized residuals:
-----------------------------------------------
<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2</th>
<th>d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>16.98</td>
<td>0.1504</td>
<td>12</td>
</tr>
<tr>
<td>sp</td>
<td>16.66</td>
<td>0.1627</td>
<td>12</td>
</tr>
</tbody>
</table>

Lagrange multiplier test:
-----------------------------------------------
<table>
<thead>
<tr>
<th>Lag 1</th>
<th>Lag 2</th>
<th>Lag 3</th>
<th>Lag 4</th>
<th>Lag 5</th>
<th>Lag 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>gm</td>
<td>-0.3586</td>
<td>-0.69005</td>
<td>-1.81246</td>
<td>-0.7803</td>
<td>0.6568</td>
</tr>
<tr>
<td>sp</td>
<td>1.3149</td>
<td>-0.06239</td>
<td>-0.04997</td>
<td>-0.1384</td>
<td>0.8093</td>
</tr>
</tbody>
</table>

TR^2 P-value F-stat P-value
| gm     | 16.28   | 1.520  | 0.2214|
| sp     | 14.64   | 1.363  | 0.2937|

> fit4=mgarch(gmsp~1,~egarch(1,1),leverage=T)
> summary(fit4)

mgarch(formula.mean = gmsp ~ 1,formula.var=~egarch(1,1),
leverage = T)

Mean Equation: gmsp ~ 1
Conditional Variance Equation: ~ egarch(1, 1)
Conditional Distribution: gaussian

-----------------------------------------------
Estimated Coefficients:
-----------------------------------------------
| Value | Std.Error | t value | Pr(>|t|) |
|-------|-----------|---------|---------|
| C(1)  | 0.009157  | 0.003196| 2.865   | 0.0043037|
| C(2)  | 0.006447  | 0.002157| 2.988   | 0.0029151|
| A(1, 1)| -0.415912 | 0.158767| -2.620  | 0.0090141|
| A(2, 2)| -0.719478 | 0.364322| -1.975  | 0.0487219|
| ARCH(1; 1, 1) | 0.219687 | 0.064281| 3.418   | 0.0006724|
| ARCH(1; 2, 2) | 0.169399 | 0.075000| 2.259   | 0.0242454|
| GARCH(1; 1, 1) | 0.954872 | 0.023042| 41.440  | 0.0000000|
| GARCH(1; 2, 2) | 0.907813 | 0.051497| 17.629  | 0.0000000|
| LEV(1; 1, 1) | -0.136060 | 0.107624| -1.264  | 0.2066189|
| LEV(1; 2, 2) | -0.374486 | 0.224256| -1.670  | 0.0954339|

-----------------------------------------------
AIC(10) = -3910.694
BIC(10) = -3866.142

Normality Test:
-----------------------------------------------
<table>
<thead>
<tr>
<th>Jarque-Bera</th>
<th>P-value</th>
<th>Shapiro-Wilk</th>
<th>P-value</th>
</tr>
</thead>
</table>
gm  28.43  6.717e-007  0.9893  0.9011985 
sp  222.96  0.000e+000  0.9762  0.0007066 

Ljung-Box test for standardized residuals:
----------------------------------------------------
                  Statistic P-value Chi^2-d.f.
gm   19.568    0.07572   12
sp   9.854     0.62873   12

Ljung-Box test for squared standardized residuals:
----------------------------------------------------
                  Statistic P-value Chi^2-d.f.
gm   19.683    0.07332   12
sp   5.395     0.94346   12

Lagrange multiplier test:
----------------------------------------------------
             Lag 1       Lag 2       Lag 3       Lag 4       Lag 5       Lag 6 
gm  -0.05071  -0.24765  -1.6586  -0.5954  0.906238 -1.718
sp  -0.23843  0.08391  -0.4111  -0.4491  0.009386 -1.073

   TR^2 P-value F-stat P-value 
gm  18.667    0.0969  1.7493  0.1475
sp  5.123     0.9538  0.4695  0.9832

*** Demonstration of forecasts
> predict(fit2,5)

$series.pred:

    , , gm
    , , sp

[1,]  0.01011233  0.006550963
[2,]  0.01011233  0.006550963
[3,]  0.01011233  0.006550963
[4,]  0.01011233  0.006550963
[5,]  0.01011233  0.006550963

$sigma.pred:

    , , gm
    , , sp

[1,]  0.1161069  0.05758898
[2,]  0.1149695  0.05642254
[3,]  0.1138566  0.05538088
[4,]  0.1127669  0.05445162
[5,]  0.1117043  0.05362331

$R.pred:

, , gm
Dynamic correlation models
Write the conditional covariance matrix as
\[ \Sigma_t = D_t R_t D_t \]
where \( D_t = \text{diag}\{\sigma_{1t}, \ldots, \sigma_{kt}\} \) is the diagonal matrix of conditional volatility of the component series and the diagonal elements of \( R_t \) is 1. In other words, \( \sigma_{it}^2 = \text{Var}(r_{it}|F_{t-1}) \) and \( R_t \) is a correlation matrix.
The elements of \( D_t \) are often obtained by univariate volatility models.
The dynamic cross-correlation (DCC) models focus on the time-evolution of the off-diagonal elements of \( R_t \).

**DCC model** by Tse and Tsui (2002):
\[ R_t = (1 - \lambda_1 - \lambda_2) R + \lambda_1 \Psi_{t-1} + \lambda_2 R_{t-1}, \]
where \( 0 \leq \lambda_1, \lambda_2 < 1 \) such that \( 0 \leq \lambda_1 + \lambda_2 < 1 \), \( R \) is \( k \times k \) positive-definite correlation matrix and \( \Psi_{t-1} \) is the \( k \times k \) sample cross-correlation matrix of some recent asset innovations, e.g., the correlation matrix of \( \{u_{t-1}, u_{t-2}, \ldots, u_{t-m}\} \) for some pre-specified positive integer \( m \), where \( u_t = D_t^{-1} a_t \) and \( a_t = r_t - \mu_t \).

**DCC model** by Engle (2002):
\[ R_t = W_t^{-1} Q_t W_t^{-1}, \]
where \( Q_t = [q_{ij,t}] \) is a \( k \times k \) positive-definite matrix, \( W_t = \text{doag}\{\sqrt{q_{11,t}}, \ldots, \sqrt{q_{kk,t}}\} \) is a normalization matrix, and

\[
Q_t = (1 - \alpha_1 - \alpha_2) \bar{Q} + \alpha_1 u_{t-1} u_{t-1}' + \alpha_2 Q_{t-1},
\]

where \( u_t = D_t^{-1} a_t \), \( \bar{Q} \) is the sample cross-correlation matrix of \( u_t \), \( \alpha_i \geq 0 \), and \( 0 \leq \alpha_1 + \alpha_2 < 1 \).

The two DCC models differ in the way by which the cross-correlations are updated. Engle’s model requires normalization because it uses \( u_{t-1} \) in the updating. The Tse and Tsui’s model, on the other hand, uses \( m \) innovations in the updating. The choice of \( m \) affects the estimated fitted cross-correlations. A larger \( m \) provides smoother cross-correlations, but the resulting correlations may not be able to show the impact of a large shock quickly.

**Multivariate Student-\( t \) distribution with \( v \) degrees of freedom**

A \( k \)-dimensional random vector \( \epsilon_t \) follows a multivariate Student-\( t \) distribution with \( v \) degrees of freedom if its probability density function in

\[
f(\epsilon) = \frac{\Gamma((v + k)/2)}{\pi^{(k/2)} \Gamma(v/2) [1 + (v - 2)^{-1} \epsilon' \epsilon]^{-(v+k)/2}}, \quad \epsilon \in R^k.
\]

The variance of each component of \( \epsilon_t \) is 1.