Solutions to Homework Assignment 1

See the associated R analysis for further details.

1. Consider the model 
\[(1 - 1.2B + 0.35B^2)Y_t = 2.5X_{t-1},\] 
where \(X_t\) is the input variable. Obtain the impulse response function of the model. Is the model stable? Why? What is the steady state gain of the model?

**Answer:** The simple way to obtain impulse response function is to do long division, namely \[\frac{2.5B}{1-1.2B+0.35B^2} = 2.5B + 3B^2 + 2.725B^3 + 2.22B^4 + \cdots.\] Alternatively, using \(\psi(B) = \frac{2.5B}{1-1.2B+0.35B^2}\), we have 
\[(1 - 1.2B + 0.35B^2)(\psi_0 + \psi_1B + \psi_2B^2 + \psi_3B^3 + \cdots) = 2.5B.\]

Simple polynomial multiplication shows that 
\[
\psi_0 + (\psi_1 - 1.2\psi_0)B + (\psi_2 - 1.2\psi_1 + 0.35\psi_0)B^2 + (\psi_3 - 1.2\psi_2 + 0.35\psi_1)B^3 + \cdots = 2.5B.
\]

Consequently, by equating the coefficients of \(B^i\) \((i = 0, 1, \ldots)\), we have
\[
\psi_0 = 0
\]
\[
\psi_1 = 2.5 + 1.2\psi_0 = 2.5
\]
\[
\psi_2 = 1.2\psi_1 - 0.35\psi_0 = 1.2 \times 2.5 = 3
\]
\[
\psi_3 = 1.2\psi_2 - 0.35\psi_1 = 1.2 \times 3 - 0.35 \times 2.5 = 2.725
\]
\[
\psi_4 = 1.2\psi_3 - 0.35\psi_2 = 1.2 \times 2.725 - 0.35 \times 3 = 2.22, \ldots
\]

The system is stable, because the solutions of \(1 - 1.2B + 0.35B^2 = (1 - 0.7B)(1 - 0.5B) = 0\) are 1/7 and 2, which are greater than 1. Finally, the steady state gain is \(2.5/(1-1.2+0.35) = 2.5/0.15 \approx 16.67.\) [The steady state gain is obtained by letting \(B = 1\) in the ratio of two polynomials.]

**Remark:** Once you are familiar with the R script CornerFun, you can use it to do the long division. In this particular case, we have omega = c(0,2.5) and delta = c(1.2,-0.35). The following output shows the solutions

```r
> omega=c(0,2.5)
> delta=c(1.2,-0.35)
> CornerFun(omega,delta,ms=5,mr=5)
Impulse responses:  0  2.5  3  2.725  2.22  1.71025  1.2753  0.9317725  0.671772  0.480006
0.340887  0.2410623  0.1699643  0.1195854  0.08401494  0.05896305  0.04135043
(output truncated)
```
2. Consider the model

\[ Y_t = 1.5 + \frac{3B^2 - 1.0B^3}{1 - 0.6B}X_t + \frac{1}{1 - 0.3B - 0.4B^2}a_t, \]

where \( a_t \) is a sequence of independent and identically distributed random variables. Compute the impulse response function of the model and derive a 5 \times 5 table of Corner method for the transfer function model.

**Answer:** Use the R script `CornerFun` to do the division and to compute the Corner table. Here the polynomial in the numerator can be represented by \( \text{omega} = (0, 0, 3, -1) \) whereas that of the denominator is \( \text{delta} = c(0.6) \). Consequently, we have

```r
> source("CornerFun.R")
> omega=c(0,0,3,-1)
> delta=c(0.6)
> CornerFun(omega,delta,ms=5,mr=5)
```

**Impulse responses:**

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</tr>
</tbody>
</table>

**Corner Table**

3. The data file `bjserm.txt` consists of two variables each with 150 observations. The data set is from the book of Box, Jenkins and Reinsel (1994). Answer the following questions:

- **Build univariate models for the two time series, including model checking.**
  **Answer:** The univariate models for \( x_t \) and \( y_t \) (column 1 and 2, respectively) are

  \[
  (1 + 0.44B)(1 - B)x_t = a_{1t}, \quad \sigma_1^2 = 0.08 \\
  (1 - 0.28B - 0.23B^2)y_t = a_{2t}, \quad \sigma_2^2 = 1.84.
  \]

  These two models are adequate as the residuals \( Q(m) \) statistics show \( Q(10) = 11.26 \) and \( 9.91 \). Comparing with \( \chi_9^2 \) and \( \chi_8^2 \) distributions, respectively, these statistics fail to reject the null hypothesis of no serial correlations.

- **Identify the Granger causality between the two variables, if any.**
  **Answer:** Using a VAR(9) model, the Granger test suggests a transfer function relation between the two series, with \( x_t \) as the input variable.

- **If there exists a unidirectional relationship, build a transfer function model that relates the two time series.**

**Answer:** Since both $x_t$ and $y_t$ have unit root, we work on the differenced series $dx_t$ and $dy_t$. Using the Corner method, one can find two possible models. The first model is

$$dy_t = 0.028 + \frac{(4.72 - 0.01B - 0.20B^2)B^3}{1 - 0.74B}dx_t + \frac{1}{1 + 0.42B}a_t.$$  

An alternative model is

$$dy_t = 0.029 + \frac{4.73B^3}{1 - 0.72B}dx_t + \frac{1}{1 + 0.41B}a_t.$$  

Model checking indicates both models are reasonable.

4. The data file `w-oilgas-9710.txt` contains the weekly prices of crude oil and regular gasoline in the U.S. The dates are also given in the file. Is there any Granger causality between the two time series? Why?

**Answer:** Based on Granger test, there is no unidirectional relation between the two series. In other words, the two series are inter-related.

5. Problem 1.2 of Chapter 1 of the textbook. Specifically, use the matrices

$$C = \begin{bmatrix} 0.8 & 0.4 \\ -0.3 & 0.6 \end{bmatrix}, \quad S = \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}$$

and the following command

```r
m2=VARMA$im(200,malags=c(1),theta=C,sigma=S)
```

```r
zt = m2$series
```

Generate 200 observations from the VMA(1) model, $z_t = a_t - Ca_{t-1}$, where $a_t$ are iid $N(0, S)$.

(a) Plot the time series $z_t$.

**Answer:** You can use the command `MTSplot` to show the plot. Output omitted.

(b) Obtain the first two lags of sample CCM of $z_t$.

**Answer:** For the particular realization I obtained, the first two CCM are

$$\rho_1 = \begin{bmatrix} -0.502 & -0.335 \\ 0.289 & -0.246 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 0.046 & -0.010 \\ -0.217 & 0.021 \end{bmatrix}.$$  

(c) Test $H_0 : \rho_1 = \cdots = \rho_5 = 0$ versus $H_a : \rho_i \neq 0$ for some $i \in \{1, \ldots, 5\}$. Draw the conclusion using the 5% significance level.

**Answer:** $Q_2(5) = 142$ with 20 degrees of freedom. The $p$-value is close to zero so that there are serial correlations in the data.