Problem A: (45 pts) Answer briefly the following questions. Each question has three points.

1. Define a temporary change in the mean for an ARMA($p,q$) process at the time index $t = h$.

   Answer: Let $I_t^{(h)}$ be the indicator for the time index $t = h$. The model is
   \[ Z_t = \frac{\omega}{1 - \delta B} I_t^{(h)} + X_t, \]
   where $Z_t$ is the observed time series and $X_t$ is the outlier-free ARMA($p,q$) process, $\omega$ is the initial change in the level, and $0 < \delta < 1$ is the discounting rate of the intervention impact.

2. Consider the model
   \[ z_t = \mu + \frac{\omega}{1 - B} I_t^{(h)} + X_t, \quad t = 1, \ldots, T, \]
   where $\mu$ is a constant, $\omega$ is a parameter, and $I_t^{(h)}$ is the indicator for the time index $h$. Suppose that $X_t$ is a Gaussian white noise series with mean zero and variance $\sigma^2$. Derive the maximum likelihood estimate of $\omega$? What is the distribution of the estimate?

   Answer: Under normality, the MLE of $\mu$ is $\bar{z}_1 = \sum_{t=1}^{h-1} z_t / (h - 1)$ and the MLE of $\mu + \omega$ is $\bar{z}_2 = \sum_{t=h}^{T} z_t / (T - h + 1)$. Using the properties of MLE, the MLE estimate of $\omega$ is $\hat{\omega} = \bar{z}_2 - \bar{z}_1$. Using properties of sample mean, $\bar{z}_1 \sim N(\mu, \sigma^2_1)$ and $\bar{z}_2 \sim N(\mu + \omega, \sigma^2_2)$, where $\sigma^2_1$ and $\sigma^2_2$ are
   \[ \sigma^2_1 = \sigma^2 / (h - 1) \]
   \[ \sigma^2_2 = \sigma^2 / (T - h + 1). \]
   Consequently, $\hat{\omega} \sim N(\omega, \sigma^2[1/(h - 1) + 1/(T - h + 1)])$, where we have used the independence of $\bar{z}_1$ and $\bar{z}_2$.

3. Consider the model of the prior question. Suppose that $X_t$ follows the MA(1) model
   \[ X_t = a_t - \theta a_{t-1}, \]
   where $a_t$ is a Gaussian white noise series with mean zero and variance $\sigma^2_a$. A natural estimate of $\omega$ is $\hat{\omega} = \bar{z}_2 - \bar{z}_1$, where $\bar{z}_1 = \sum_{t=1}^{h-1} z_t / (h - 1)$ and $\bar{z}_2 = \sum_{t=h}^{T} z_t / (T - h + 1)$. What is the distribution of this estimate?

   Answer: The key idea here is to work out the variances of $(h - 1)\bar{z}_1$ and $(T - h + 1)\bar{z}_2$ under the MA(1) assumption. The distribution of $\hat{\omega}$ will be normal under the normality
assumption. It turns out that

\[ \text{Var}[\bar{z}_1] = \frac{(1 + \theta^2)\sigma_a^2 + (1 - \theta)^2(h - 2)\sigma_a^2}{(h - 1)^2} \]

\[ \text{Var}[\bar{z}_2] = \frac{(1 + \theta^2)\sigma_a^2 + (1 - \theta)^2(T - h)\sigma_a^2}{(T - h + 1)^2}. \]

In addition, the covariance between \( \bar{z}_1 \) and \( \bar{z}_2 \) is \(-\theta\sigma_a^2/[(h - 1)(T - h + 1)]\). Therefore, \( \hat{\omega} \) is given by

\[ \frac{(1 + \theta^2)\sigma_a^2 + (1 - \theta)^2(h - 2)\sigma_a^2}{(h - 1)^2} + \frac{(1 + \theta^2)\sigma_a^2 + (1 - \theta)^2(T - h)\sigma_a^2}{(T - h + 1)^2} + \frac{2\theta\sigma_a^2}{(h - 1)(T - h + 1)}. \]

4. Consider the estimate \( \hat{\omega} \) of the prior problem. Let \( h = [T/2] \), which is the integer part of \( T/2 \). Derive the limiting distribution of \( \sqrt{T}\hat{\omega} \) as \( T \to \infty \) when (a) \( \theta = 0.9 \) and (b) \( \theta = 0 \). Discuss the impact of \( \theta \) on the limiting distribution.

Answer: From the prior solution, the asymptotic variance of \( \sqrt{T}\hat{\omega} \) is

\[ (1 - \theta)^2\sigma_a^2T\left(\frac{h - 2}{(h - 1)^2} + \frac{T - h}{(T - h + 1)^2}\right) \to 4(1 - \theta)^2\sigma_a^2. \]

Therefore, for case (a): \( \theta = 0.9 \), \( \sqrt{T}\hat{\omega} \sim N(\omega, 4(1 - 0.9)^2\sigma_a^2) \). For case (b): \( \theta = 0 \), \( \sqrt{T}\hat{\omega} \sim N(\omega, 4\sigma_a^2) \). From the results, \( \theta \) plays an important role in determining the limiting distribution of \( \sqrt{T}\hat{\omega} \).

5. Consider the data \( \{r_1, r_2, \ldots, r_n\} \) from the model \( r_t = \mu + e_t \), where \( e_t = \sigma_t a_t \) with \( a_t \) being a Gaussian white noise with variance \( \sigma_a^2 \) and \( \sigma_a^2 = \alpha_0 + \alpha_1 e_{t-1}^2 \). Write down the conditional likelihood function of the data given \( r_1 \).

Answer: Let \( F_{t-1} \) be the \( \sigma \)-field generated by \( r_{t-1}, r_{t-2}, \ldots, r_1 \). The conditional likelihood function is

\[ f(r_1, \ldots, r_n|r_1) = \prod_{t=2}^{T} f(r_t|F_{t-1}) \]

\[ = \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{(r_t - \mu)^2}{2\sigma_t^2}\right]. \]

6. Consider two time series

\[ X_t = 0.4X_{t-1} + a_t, \]

\[ Y_t = 1.2Y_{t-1} - 0.32Y_{t-2} + b_t \]

where \( \{a_t\} \) and \( \{b_t\} \) are two independent white noises with unit variance. What is the model ARMA model of \( Z_t = X_t + Y_t \)?
10. Let $r_t$ be the daily log return of an asset with mean zero. Describe a method to test the conditional heteroscedasticity in $r_t$. What is the test statistic? What is the asymptotic distribution of the test?

Answer: Consider the multiple linear regression $r_t^2 = \alpha_0 + \sum_{i=1}^{k} \alpha_i r_{t-i}^2 + \epsilon_t$. A test can be constructed by testing $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ versus $H_a : \alpha_i \neq 0$ for some $i$. The usual $F$-statistic can be used. The asymptotic distribution is chi-square with $k$ degrees of freedom.

7. Consider an AR(1) model $z_t = \phi z_{t-1} + a_t$, where $a_t$ is a white noise with variance $\sigma^2_a$. Suppose that among the 200 observations \{z_1, \ldots, z_{200}\}, z_{100}$ and $z_{101}$ are missing. Derive the conditional distribution of $(z_{100}, z_{101})$ given the model and the data.

Answer: From the AR(1) model, the two missing values are related to $z_{99}, z_{100}, z_{101}, z_{102}$. More specifically, we can obtain

$$
\begin{bmatrix}
\phi z_{99} \\
0 \\
z_{102}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
\phi & -1 & 0 \\
0 & \phi & \phi
\end{bmatrix}
\begin{bmatrix}
z_{100} \\
z_{101} \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
-a_{100} \\
a_{101} \\
a_{102}
\end{bmatrix}.
$$

This is a multiple linear regression with 3 observations and two unknown parameters. Express the regression as $Y = X\beta + E$, where $\beta = (z_{100}, z_{101})^t$. The estimate of $\beta$ is $\beta = (X'X)^{-1}(X'Y)$, which is distributed as $N(\beta, \sigma^2_a(X'X)^{-1})$.

8. Consider an ARMA($p, d, q$) process $z_t$ satisfying $\phi(B)(1-B)^d z_t = \theta(B)a_t$, where $a_t$ is a Gaussian white noise series with mean zero and variance $\sigma^2_a$. Suppose that the observed data is $y_t = \frac{\theta(B)}{\phi(B)(1-B)^d}(a_t + \omega I_{t(h)})$, where $I_{t(h)}$ is the indicator for time index $h$. Given the model, derive an estimate of $\omega$? What is the distribution of the estimate?

Answer: $\omega$ is the size of an innovational outlier. It is estimated by the residual at time $t = h$. Thus, letting $e_t = [(1-B)^d\phi(B)/\theta(B)]y_t$, $\hat{\omega} = e_{h}$. Under the null hypothesis of no outlier, $\hat{\omega}$ is normal with mean zero and variance $\sigma^2_a$.

9. Consider the ARMA(1,1) model $X_t = 0.8X_{t-1} + a_t - 0.4a_{t-1}$, where $a_t$ is a Gaussian white noise with mean zero and variance 1. Let $Z_t = X_{2t} + X_{2t-1}$ for $t \geq 1$. What is the order of the ARMA model for $Z_t$?

Answer: $Z_t$ is a 2-aggregate process. Applying $(1-0.8^2B)Z_t$, we see that $Z_t$ follows an ARMA(1,1) model.

10. Let $r_t$ be the daily log return of an asset with mean zero. Describe a method to test the conditional heteroscedasticity in $r_t$. What is the test statistic? What is the asymptotic distribution of the test?

Answer: Consider the multiple linear regression $r_t^2 = \alpha_0 + \sum_{i=1}^{k} \alpha_i r_{t-i}^2 + \epsilon_t$. A test can be constructed by testing $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ versus $H_a : \alpha_i \neq 0$ for some $i$. The usual $F$-statistic can be used. The asymptotic distribution is chi-square with $k$ degrees of freedom.
11. Consider the model
\[ r_t = 0.01 + e_t, \quad e_t = \sigma_t a_t, \quad a_t \sim \text{iid } N(0, 1) \]
\[ \sigma_t^2 = 0.01 + 0.85\sigma_{t-1}^2 + 0.1e_{t-1}^2. \]
Suppose that \( r_{100} = 0.1 \) and \( \sigma_{100}^2 = 0.09 \). Compute the 1-step ahead forecast of \( r_t \) at the forecast origin \( t = 100 \)? What is the associated volatility forecast?

Answer: \( r_{100}(1) = 0.01 \). Also, \( \epsilon_{100} = 0.1 - 0.01 = 0.09 \). Thus, \( \sigma_{101}^2 = 0.01 + 0.85(0.09) + 0.1(0.09)^2 = 0.0873 \). The volatility forecast is \( \sqrt{0.0873} = 0.295 \).

12. What is the impact when the serial correlations in the residuals of a linear regression model are overlooked?

Answer: Incorrect estimation of the variances of coefficient estimates. In other words, the \( t \)-ratios are not reliable.

13. Suppose that \( X_t \) follows the AR(1) model \( X_t = 0.9X_{t-1} + a_t \), where \( a_t \) is a Gaussian white noise series with mean zero and variance \( \sigma_a^2 \). What is the model for the overdifferenced series \( Z_t = (1 - B)X_t \)?

Answer: \( Z_t = (1 - B)X_t = (1 - B) \frac{1}{1 - 0.9B} a_t \) so that \( (1 - 0.9B)Z_t = (1 - B)a_t \), which is a non-invertible ARMA(1,1) model.

14. Describe two sources by which a GARCH model can introduce excess kurtosis.

Answer: (a) The excess kurtosis of the innovation \( a_t \) and (b) the dynamic of the GARCH structure.

15. Let \( X_t \) be the daily price range, (High minus low), of a particular stock. Assume that \( X_t/\psi_t = \epsilon_t \), where \( \epsilon_t \) is an exponential distribution with mean 1 and \( \psi_t = 0.1 + 0.9\psi_{t-1} + 0.05X_{t-1} \). What is the \( E(X_t) \)? What is \( \text{Var}(X_t) \)?

Answer: Since \( E(X_t) = E(\psi_t \epsilon_t) = E(\psi_t) \) and using stationarity, we have \( E(\psi_t) = 0.1 + 0.9E(\psi_{t-1}) + 0.05E(X_{t-1}) \). Therefore, \( E(X_t) = 0.1/(1 - 0.9 - 0.05) = 2 \). Using the result of Lec15-08, we have \( \text{Var}(X_t) = \frac{2^2(1 - 0.9^2 - 2 \cdot 0.05)(0.9)}{1 - 2(0.05)^2 - 2 \cdot 0.05(0.9)} = 4.21 \).

**Problem B.** (20 pts) Suppose that the univariate time series \( z_t \) follows the model
\[ z_t = z_{t-1} + y_t, \quad y_t = (1 - \theta_1 B - \theta_2 B^2)a_t, \quad t = 1, \ldots, T, \]
where \( a_t \) is a Gaussian white noise with mean 0 and variance \( \sigma_a^2 \) and the MA model is invertible. Derive the limiting distributions of the following statistics as \( T \to \infty \):

1. (5 points) \( T^{-2} \sum_{t=1}^{T} z_{t-1}^2 \)

   Answer: Based on the Theorem of Lec11, the key quantity to consider is \( \sigma^2 = \lim_{T \to -\infty} E(T^{-1}S_T^2) \). Since \( y_t = (1 - \theta_1 B - \theta_2 B^2)a_t \), it is easy to calculate that (a) \( \sigma_y^2 = (1 + \theta_1^2 + \theta_2^2)\sigma_a^2 \) and (b) \( \sigma^2 = \sigma_y^2(1 + 2\rho_1 + 2\rho_2) = \sigma(1 - \theta_1 - \theta_2)^2 \). Therefore,
\[ T^{-2} \sum_{t=1}^{T} z_{t-1}^2 \Rightarrow \sigma_a^2(1 - \theta_1 - \theta_2)^2 \int_0^1 W(r)^2 dr, \]
where $W(.)$ denotes the standard Brownian motion.

2. (5 points) $T^{-1} \sum_{t=1}^{T} z_{t-1} y_t$
   
   Answer: $T^{-1} \sum_{t=1}^{T} z_{t-1} y_t \Rightarrow \frac{\sigma_i^2(1-\theta_1-\theta_2)^2}{2} \left( W(1)^2 - \frac{1+\theta_1^2+\theta_2^2}{(1-\theta_1-\theta_2)^2} \right)$. 

3. (5 points) $T(\hat{\pi} - 1)$, where $\hat{\pi}$ is the least squares estimate of the AR(1) model $z_t = \pi z_{t-1} + \epsilon_t$.
   
   Answer: $T(\hat{\pi} - 1) \Rightarrow \frac{(1/2)(W(1)^2-(1+\theta_1^2+\theta_2^2)/(1-\theta_1-\theta_2)^2)}{\int_0^1 W(r)^2 dr}$. 

4. (5 points) The $t$-ratio for testing $H_0 : \pi = 1$ vs $H_a : \pi < 1$.
   
   Answer: $t_\pi \Rightarrow \frac{|1-\theta_1-\theta_2|}{\sqrt{1+\theta_1^2+\theta_2^2}} \left( \frac{1}{\sigma_i^2} W(1)^2 - \frac{1+\theta_1^2+\theta_2^2}{(1-\theta_1-\theta_2)^2} \right) \int_0^1 W(r)^2 dr \right)^{1/2} \frac{1}{\sigma_i^2}$. 

Problem C. (15 pts) Consider a time series $z_t = T_t + S_t + \epsilon_t$, where the components follow the models

$$(1-B)^2 T_t = \epsilon_{1t}, \quad (1 + B + B^2 + B^3) S_t = \epsilon_{2t}, \quad (1 - 0.5B) \epsilon_t = \epsilon_{3t},$$

where $\epsilon_{it}$ are independent white noise series with mean zero and variance $\sigma_i^2$, $i = 1, 2, 3$.

Answer the following questions:

1. (3 points) Write down a state-space model for $T_t$.
   
   Answer: The observation eq. is $T_t = [1, 0] S_t$, whereas the state-transition eq. is

   $$\begin{bmatrix} T_t \\ T_{t-1} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_{t-1} \\ T_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \epsilon_{1t}.$$ 

2. (3 points) Write down a state-space model for $S_t$.
   
   Answer: The observation eq. is $S_t = [1, 0] X_t$, whereas the state-transition eq. is

   $$\begin{bmatrix} S_t \\ S_{t-1} \\ S_{t-2} \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} S_{t-1} \\ S_{t-2} \\ S_{t-3} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \epsilon_{2t}.$$ 

3. (3 points) Write down a state-space model for $\epsilon_t$.
   
   Answer: The observation eq. is $\epsilon_t = \epsilon_t$ and the state-transition eq. is $\epsilon_t = 0.5 \epsilon_{t-1} + \epsilon_{3t}$. 

4. (6 points) Write down a state-space model for $z_t$. 

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Answer: Let \( S_t = (T_t, T_{t-1}, S_t, S_{t-1}, S_{t-2}, e_t)' \) and \( \epsilon_t = (\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t})' \). We have the observation eq. \( z_t = (1, 0, 1, 0, 0, 1)S_t \) and the state-transition eq. is \( S_t = FS_{t-1} + G\epsilon_t \), where

\[
F = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad G = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

**Problem D.** (15 pts) Consider the daily log returns, in percentages, of the stock of Adobe Systems, Inc. from 1997 to 2006. Analysis of the series, including GARCH modelling, is given in the attached output. Use the output to answer the following questions.

1. (3 pts) Ignoring the conditional heteroscedasticity, we saw a significant Ljung-Box statistic for the log return series. Write down the fitted AR(1) model.
   Answer: \((1 + 0.051B)(r_t - 0.083) = a_t\), where \(\sigma_a^2 = 12.04\).

2. (3 pts) Is there any ARCH effect in the residuals of the AR(1) model? Why?
   Answer: Yes, the Ljung-Box Q statistic of the squared residuals shows \(Q(10) = 142.52\) with p-value close to 0.

3. (3 pts) Write down the fitted Gaussian GARCH(1,1) model.
   Answer: Mean equation is \(r_t = 0.115 + e_t\), \(e_t = \sigma_t a_t\) with \(a_t\) being iid \(N(0,1)\). The volatility equation is \(\sigma_t^2 = 0.034 + 0.035e_{t-1}^2 + 0.963\sigma_{t-1}^2\).

4. (3 pts) Write down the fitted GARCH(1,1) model with Student-\(t\) innovations.
   Answer: Mean equation is \(r_t = 0.041 + e_t\), \(e_t = \sigma_t a_t\) with \(a_t\) being a standardized Student-\(t\) distribution with 4.91 degrees of freedom. The volatility equation is \(\sigma_t^2 = 0.008 + 0.018e_{t-1}^2 + 0.981\sigma_{t-1}^2\).

5. (3 pts) Compare the two fitted GARCH(1,1) model. Which one is preferred? Why?
   Answer: The GARCH(1,1) model with Student-\(t\) innovation is preferred because it has a high log likelihood value and allows for heavy tails.