1. Taking the first difference, we have

\[(1 - B)Z_t = (1 - B)X_t + (1 - B)b_t = a_t + b_t - b_{t-1}.\]

Let \(W_t = a_t + b_1 - b_{t-1}\). It is easy to see that (a) \(\text{Var}(W_t) = 3\), (b) \(\text{Cov}(W_t, W_{t-1}) = -1\), and (c) \(\text{Cov}(W_t, W_{t-j}) = 0\) for \(j > 1\). Thus, \(W_t\) is an MA(1) series, which can be written as \(W_t = \epsilon_t - \theta \epsilon_{t-1}\) with \(\{\epsilon_t\}\) being an i.i.d. sequence with mean zero and variance \(\sigma_e^2\). The values of \(\theta\) and \(\sigma_e^2\) can be determined by equating the variance and lag-1 covariance of \(W_t\). Specifically, from the MA(1) model, we have \(\text{Var}(W_t) = (1 + \theta^2)\sigma_e^2\) and \(\text{Cov}(W_t, W_{t-1}) = -\theta \sigma_e^2\). Consequently, we have \((1 + \theta^2)\sigma_e^2 = 3\) and \(\theta \sigma_e^2 = 1\). Taking the ratio, we have \(\theta/(1 + \theta^2) = 1/3\). The solutions for \(\theta\) are approximately 0.382 and 2.618. Since we are interested in invertible MA(1) model, \(\theta\) must satisfy the condition \(|\theta| < 1\). Therefore, \(\theta = 0.382\) and \(\sigma_e^2 = 2.618\).

2. I use R to perform the simulation. For simplicity, I do not show the plot, but give the actual values of sample ACF.

(a)

```r
> at=rnorm(200)
> zt=rep(0,200)
> zt[1]=at[1]
> for (i in 2:200){
+ zt[i]=zt[i-1]+at[i]
+ }
> plot(zt,type='l')
> m1=acf(zt,lag=12)
> print(m1$acf,digits=2)

[,1]
[1,] 1.00
[2,] 0.95
[3,] 0.90
[4,] 0.86
[5,] 0.81
[6,] 0.76
[7,] 0.72
[8,] 0.68
```
Alternatively, you may use the command ‘cumsum’, cumulative sum.

```r
> yt = cumsum(at)
> m2 = acf(yt, lag=12)
> print(m2$acf, digits=2)

[,1]
[1,] 1.00
[2,] 0.95
[3,] 0.90
```

(b) (Note: the slope dominates the plot.)

```r
> at = rnorm(200)
> for (i in 2:200)
+   zt[i] = zt[i-1] + 5 + at[i]
> plot(zt, type='l')
> m2 = acf(zt, lag=12)
> print(m2$acf, digits=2)

[,1]
[1,] 1.00
[2,] 0.98
[3,] 0.97
[4,] 0.95
[5,] 0.94
[6,] 0.92
[7,] 0.91
[8,] 0.89
[9,] 0.88
[10,] 0.86
[11,] 0.85
[12,] 0.83
[13,] 0.82
```

(c) Seasonal model

```r
> zt = arima.sim(200, model=list(ar=c(0, 0, 0, 0.8)), sd=1)
> plot(zt, type='l')
> m3 = acf(zt, lag=12)
> print(m3$acf, digits=2)

[,1]
[1,] 1.000
```

2
(d) Complex unit roots (You should see the sine and cosine pattern of ACFs).

```r
> at=rnorm(200)
> zt[1]=at[1]
> for (i in 3:200){
+   zt[i]=sqrt(3)*zt[i-1]-zt[i-2]+at[i]
+ }
> plot(zt,type='l')
> m4=acf(zt,lag=12)
> print(m4$acf,digits=2)
```

(e) An approximate model for fractional difference time series.
You should see small, but significant, ACFs.

```r
> zt=arima.sim(1000,model=list(ar=c(0.9),ma=c(-0.8)),sd=1)
> plot(zt,type='l')
> m5=acf(zt,lag=12)
> print(m5$acf,digits=2)
```
3. Applying \((1 - 0.8B)\) to \(Z_t\), we have

\[
(1 - 0.8B)Z_t = (1 - 0.8B)X_t + (1 - 0.8B)Y_t = a_t - 0.3a_{t-1} + b_t.
\]

Let \(W_t = a_t - 0.3a_{t-1} + b_t\). Similar to Q1, \(W_t\) is an MA(1) series and can be written as \(W_t = e_t - \theta e_{t-1}\) with \(e_t\) being a Gaussian white noise series with mean zero and variance \(\sigma^2_e\). Working out the algebra, we have approximately \(\theta = 0.147\) and \(\sigma^2_e = 6.82\).

4. Suppose that \(X_t\) follows the model \(X_t = 0.8X_{t-1} + a_t\), where \(a_t\) is a Gaussian white noise series with mean 0 and variance 1. What is the model for the aggregated series \(Y_t = X_{2t} + X_{2t-1}\)?

Applying \((1 - 0.8^2B)\) to \(Y_t\) and noting that “B” of \(Y_t\) is \(B^2\) of \(X_t\), we have

\[
(1 - 0.64B)Y_t = (X_{2t} + X_{2t-1}) - 0.64(X_{2t-2} + X_{2t-3})
\]

\[
= (X_{2t} - 0.64X_{2t-2}) + (X_{2t-1} - 0.64X_{2t-3})
\]

\[
= [(X_{2t} - 0.8X_{2t-1}) + 0.8(X_{2t-1} - X_{2t-2})] + [(X_{2t-1} - 0.8X_{2t-2}) + 0.8(X_{2t-2} - X_{2t-3})]
\]

\[
= a_{2t} + 0.8a_{2t-1} + a_{2t-1} + 0.8a_{2t-2}
\]

\[
= a_{2t} + 1.8a_{2t-1} + 0.8a_{2t-2}.
\]

In the time scale of \(Y_t\), the right hand side of the previous equation involves 1 lag so that it is an MA(1) model. Consequently, \(Y_t\) follows an ARMA(1,1) model with AR polynomial \((1 - 0.64B)\). The MA part can be obtained as before.

5. For part (1), write the model as \(X_t = a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}\). Simply computing the autocovariances at lags 0, 1, 11, 12, and 13, you can obtain the results given in the lecture note. For part (2), write the model as \(X_t = a_t - \theta a_{t-1} - \Theta a_{t-12}\). The result given in the lecture note follows by direct computing the autocovariance of lags 0, 1, 11, and 12. All other lags of ACFs are zero.