1. (a) Akaike’s approach: The dimension of state vector is \( m = \max\{2, 1+1\} = 2 \) so that \( S_t = (z_t, z_{t+1})' \) and, from the model, \( \psi_1 = 1.8 \). Thus,

\[
\begin{bmatrix}
  z_{t+1} \\
  z_{t+2|t+1} \\
  z_t
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  -0.4 & 1.3 \\
  1 & 1.8
\end{bmatrix}
\begin{bmatrix}
  z_t \\
  z_{t+1|t} \\
  a_{t+1}
\end{bmatrix}
\]

Aoki’s approach: The state vector is \( S_t = (z_{t-1}, z_{t-2}, a_{t-1})' \) and the model is

\[
\begin{bmatrix}
  z_t \\
  z_{t-1} \\
  a_t
\end{bmatrix} = \begin{bmatrix}
  1.3 & -0.4 & 0.5 \\
  1 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  z_{t-1} \\
  z_{t-2} \\
  a_{t-1}
\end{bmatrix} + \begin{bmatrix}
  1 \\
  0 \\
  1
\end{bmatrix} a_t
\]

Harvey’s approach: The dimension of the state vector is \( m = \max\{2\} = 2 \) and the state vector is \( S_t(z_{t|t-1}, z_{t+1|t-1})' \). The model becomes

\[
\begin{bmatrix}
  z_{t+1|t} \\
  z_{t+2|t} \\
  z_t
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  -0.4 & 1.3 \\
  1 & 1.94
\end{bmatrix}
\begin{bmatrix}
  z_{t-1} \\
  z_{t+1|t-1} \\
  a_t
\end{bmatrix}
\]

2. There are several ways to derive the result that the distribution of \( X \) given \( Y = y \) is multivariate normal with mean \( \mu_x + \Sigma_{xy} \Sigma_{yy}^{-1}(y - \mu_y) \) and covariance matrix \( \Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx} \). See, for instance, the multivariate statistics book by Johnson and Wichern (2007, 6th ed.) One approach is to use density function. Here one makes use of the identities: (a) \( |\Sigma| = |\Sigma_{yy}| |\Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx}| \), where \( \Sigma \) is the covariance matrix of \((X', Y')'\). (b) \( [(X - \mu_x)', (Y - \mu_y)'] \Sigma^{-1} [(X - \mu_x)', (Y - \mu_y)']' = [X - \mu_x - \Sigma_{xy} \Sigma_{yy}^{-1} (Y - \mu_y)] [(\Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx})^{-1} [X - \mu_x - \Sigma_{xy} \Sigma_{yy}^{-1} (Y - \mu_y)] + (Y - \mu_y) \Sigma_{yy}^{-1} (Y - \mu_y) \).

These two identities allow us to partition the joint density function of \( X \) and \( Y \) into the marginal density of \( Y \) and the conditional density of \( X \) given \( Y = y \).
3. The conditional distribution of $X$ given $Y = y$ and $Z = z$ is multivariate normal with mean $\mu_x + \Sigma_{xy} \Sigma_{yy}^{-1} (y - \mu_y) + \Sigma_{xz} \Sigma_{zz}^{-1} (z - \mu_z)$ and covariance matrix $\Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx} - \Sigma_{xz} \Sigma_{zz}^{-1} \Sigma_{zx}$. This result can be obtained by using result of Problem 2 and the independence between $Y$ and $Z$ under normality.

4. Among the equations (6) to (10), the only one that needs modification is (10), which becomes

$$C_{t+1|t} = H P_{t+1|t} + W.$$  

Eq. (11) and (12) becomes

$$S_{t+1|t+1} = S_{t+1|t} + \left[ P_{t+1|t} H' + W \right] [H P_{t+1|t} H' + R]^{-1} (Z_{t+1} - Z_{t+1|t})$$

$$P_{t+1|t+1} = P_{t+1|t} - \left[ P_{t+1|t} H' + W \right] [H P_{t+1|t} H' + R]^{-1} (H P_{t+1|t} + W).$$

5. Again, there are multiple solutions. One of them is given below.

$$\begin{bmatrix}
\beta \\
z_t \\
z_{t-1}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0.5 & 0.24 \\
0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
\beta \\
z_{t-1} \\
z_{t-2}
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} a_t.$$  

$$y_t = [x_t, 1, 0] S_t.$$