Consider a two-period example of the dynamic model described in "Efficiency of Competitive Search under Asymmetric Information" with the following simplifying specifications:

1. match-specific disutility shock $\theta \sim U[0, 1]$
2. matching function $\mu(\gamma) = \min\{\gamma, 1\}$
3. no discounting: $\beta = 1$
4. zero instantaneous utility to unemployed: $b = 0$

I want to illustrate that the competitive equilibrium is generically constrained efficient, that is it coincides with the social optimal allocation only for a specific initial unemployment rate.

1 Competitive Equilibrium

I first characterize the competitive equilibrium allocation $\hat{\theta}^{CE}_1$, $\gamma^{CE}_1$, $U^{CE}_1$, $\hat{\theta}^{CE}_2$, $\gamma^{CE}_2$, $U^{CE}_2$ where $\hat{\theta}^{CE}_t$ is the hiring margin at period $t$ (workers are hired if and only if they

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draw a type $\theta_t \leq \hat{\theta}_t^{CE}$), $\gamma_t^{CE}$ is the tightness of the market at period $t$ and $U_t^{CE}$ is the continuation utility for unemployed workers at the beginning of period $t$.

Specializing Proposition 10 in the paper to the finite horizon case, I can solve for the competitive allocation. The important feature of the competitive equilibrium is that at each point in time, agents take as given the future. In particular, in the two-period example, agents at time 1 take as given future equilibrium contracts and market tightness, that is $\hat{\theta}_2^{CE}$ and $\gamma_2^{CE}$, which determine $U_2^{CE}$. Equations (1) and (2) below show that $U_2^{CE}$ affects equilibrium job creation both at time 2 and at time 1. An externality arises because agents do not take into account the impact of $U_2^{CE}$ on job creation at time 1, represented simply by $\hat{\theta}_1^{CE}$ in the example ($\gamma_t^{CE} = 1$ for any $t$). In the next subsection, I will characterize the social optimum. I will show that the two relations (1) and (2) must be satisfied also by a socially optimal allocation, but the social planner will internalize the impact of $U_2^{CE}$ on $\hat{\theta}_1^{CE}$, generically improving upon the equilibrium.

To characterize the competitive equilibrium allocation, I use Proposition 11 in the paper which shows how $\gamma$ and $\hat{\theta}$ at each period must maximize the expected utility of unemployed workers subject to a single constraint which comes from the combination of the incentive-compatibility constraints, the participation constraints and the free-entry condition (which is also substituted into the objective function). Going backward:

**Period 2.** Given that this is the last period of the economy, agents take as given that the continuation utility of being unemployed at the end of the period is going to be 0. Then, the equilibrium $\hat{\theta}_2^{CE}$ and $\gamma_2^{CE}$ solve

$$U_2^{CE} = \max_{\hat{\theta}_2, \gamma_2} \gamma_2 \left[ y\hat{\theta}_2 - \frac{\hat{\theta}_2^2}{2} \right] - \gamma_2 k,$$

s.t.

$$\gamma_2 \left[ y\hat{\theta}_2 - \frac{\hat{\theta}_2^2}{2} \right] \geq \gamma_2 k.$$

For any $k > 0$ the constraint is binding. Assume that at the solution $\gamma_2 = 1$ (which will be the case when $k$ is not too big), and the equilibrium $\hat{\theta}_2^{CE}$ solves

$$y\hat{\theta}_2^{CE} - \left( \hat{\theta}_2^{CE} \right)^2 = k.$$

Substituting in the objective, one finds the following relation between equilibrium
job creation and expected utility of being unemployed:

\[ \hat{\theta}_2^{CE} = \sqrt{2U_2^{CE}} \]  

(1)

Trivially, the expected utility of being employed, net from disutility and wages, is zero, that is \( V_2^{CE} = 0 \).

**Period 1.** In period 1, agents take as given the continuation utility of being unemployed in the next period, that is \( U_2^{CE} \). Then, the equilibrium \( \hat{\theta}_1 \) and \( \gamma_1 \) solve

\[
U_1^{CE} = \max_{\hat{\theta}_1, \gamma_1} \left[ (y - U_2^{CE}) \hat{\theta}_1 - \frac{1}{2} \hat{\theta}_1^2 \right] - \gamma_1 k + U_2^{CE}
\]

s.t.

\[
\gamma_1 \left[ (y - U_2^{CE}) \hat{\theta}_1 - \hat{\theta}_1^2 \right] \geq \gamma_1 k.
\]

Assume that at the solution \( \gamma_1^{CE} = 1 \) (which will be the case when \( k \) is not too big), then, as long as \( k > 0 \), the constraint is binding and \( \hat{\theta}_1^{CE} \) satisfies

\[
(y - U_2^{CE}) \hat{\theta}_1^{CE} - \left( \hat{\theta}_1^{CE} \right)^2 = k.
\]

Substituting in the objective function I get the relation:

\[
\hat{\theta}_1^{CE} = \sqrt{2 \left( U_1^{CE} - U_2^{CE} \right)}.
\]  

(2)

Moreover the equilibrium \( V_1^{CE} \) and \( u_2^{CE} \) for a given \( u_1 \) are

\[
V_1^{CE} = \beta s U_2^{CE} + \beta (1 - s) V_2^{CE}
\]

and

\[
u_2^{CE} = u_1 (1 - \hat{\theta}_1^{CE}) + s (1 - u_1).
\]

## 2 Social Planner

I study the Pareto frontier for a given initial rate of unemployment \( u_1 \). The Social Planner, for a given \( u_1 \), maximizes the *ex-ante* utility of unemployed workers at time 1, \( U_1 \), subject to the *ex-ante* utility of employed workers above some given value, that is \( V_1 \geq \bar{V} \) and subject to the incentive compatibility constraints, the participation
constraints and the resource constraint of the economy. Following section 5.2 in the paper, I analyze the dual version of the problem written in a recursive form. At each period in time, the planner maximize the expected stream of resources subject to two promise keeping constraint respectively for the unemployed and the employed workers, non-negativity constraints for consumption values, the incentive compatibility and the participation constraints. Going backward:

**Period 2.** Assume that at the solution $\hat{\theta}_2$ is interior, then the planner problem at the beginning of period 2 can be written as

$$P_2(V_2, U_2, u_2) = \max_{\gamma_2, \theta_2, \omega_2, C_2^V, C_2^U} u_2 \gamma_2 \left[F(\hat{\theta}_2) (y - w_2) - (1 - F(\hat{\theta}_2))w_2\right]$$

$$- u_2 (1 - \gamma_2) C_2^U - (1 - u_2) C_2^V - u_2 \gamma_2 k$$

s.t.

$$U_2 = \gamma_2 F(\hat{\theta}_2) w_2 - \gamma_2 \frac{1}{2} \theta_2^2 + \gamma_2 (1 - F(\hat{\theta}_2))w_2 + (1 - \gamma_2) C_2^U$$

$$V_2 = C_2^V$$

$$C_2^U \geq 0; C_2^V \geq 0; w_2 \geq 0; w_2 \geq 0$$

$$0 \leq \gamma_2 \leq 1$$

$$w_2 - w_2 = \hat{\theta}_2$$

where $w_2$ is the consumption given to matched workers who are hired at the end of time 2, $\omega_2$ is the consumption given to matched workers who are not hired, $C_2^U$ is the consumption given to unmatched workers and $C_2^V$ is the consumption given to workers who are employed at the beginning of the period. The last equation comes from the incentive compatibility constraints: the worker of type $\hat{\theta}_2$ has to be indifferent between working and getting $w_2 - \hat{\theta}_2$ or reporting any type $\theta_2 > \hat{\theta}_2$ and getting $\omega_2$. Notice that the consumption $w_2$ has to be constant for all $\theta_2 < \hat{\theta}_2$, otherwise they would all report the type with the highest consumption.

Assume that at the optimum $\gamma_2 = 1$. If $U_2$ is low enough, that is $U_2 < \frac{1}{2} y_2^2$ then the non-negativity constraints for $w_2$ and $C_2^U$ are binding. I can substitute for $w_2$ and get

$$\hat{\theta}_2 = \sqrt{2U_2}$$

(3)
which is exactly the same relation (1) that holds for the competitive equilibrium allocation (with a general $U_2$). Moreover,

$$P_2 (V_2, U_2, u_2) = u_2 \left[ y\sqrt{2U_2 - 2U_2} \right] - (1 - u_2) V_2 - u_2 k.$$  

Remark: if we have only one period it is straightforward to check that $\hat{\theta}_2^{CE}$ solves the planner problem for the promised utility $U_2^{CE}$.

**Period 1.** Assume that $\hat{\theta}_1$ is an interior solution, then the planner problem at the beginning of period 1 can be written as

$$P_1 (V_1, U_1, u_1) = \max_{U_1, V_1, C_1, w_1, \hat{\theta}_1} \left\{ u_1 \gamma \left[ F(\hat{\theta}_1) (y - w_1) - (1 - F(\hat{\theta}_1)) w_1 \right] \right\}$$

$$- u_1 (1 - \gamma) C_1^U - (1 - u_1) C_1^V - u_1 \gamma_1 k + P_2 (V_2, U_2, u_2)$$

s.t.

$$U_1 = \gamma_1 F(\hat{\theta}_1) (w_1 + V_2) - \gamma_1 \frac{1}{2} \hat{\theta}_1^2 + (1 - \gamma_1 F(\hat{\theta}_1)) (w_1 + U_2)$$

$$V_1 = C_1^V + sU_2 + (1 - s) V_2$$

$$u_2 = u_1 (1 - \gamma_1 F(\hat{\theta}_1)) + s (1 - u_1)$$

$$C_1^U \geq 0; C_1^V \geq 0; w_1 \geq 0; w_1 \geq 0$$

$$0 \leq \gamma_1 \leq 1$$

$$w_1 - w_1 + (V_2 - U_2) = \hat{\theta}_1$$

where $w_1$, $w_1$, $C_1^U$ and $C_1^V$ are the analog of the previous period consumption values. The last equation comes again from the incentive compatibility constraints: the worker of type $\hat{\theta}_1$ has to be indifferent between working and getting $w_1 - \hat{\theta}_1 + V_2$ or reporting any type $\theta_1 > \hat{\theta}_1$ and getting $w_1 + U_2$, where $V_2$ and $U_2$ are the promised utility respectively to employed (net from wages and disutility) and unemployed workers.

Now, suppose that at the solution $\gamma_1 = 1$ and that the non-negativity constraints for $C_1^U$ and $w_1$ are binding. Then the problem can be rewritten as

$$P_1 (V_1, U_1, u_1) = \max_{U_1, V_1, C_1, w_1, \hat{\theta}_1} \left\{ u_1 \left[ \hat{\theta}_1 \left( y - \hat{\theta}_1 - (U_2 - V_2) \right) \right] \right\}$$

$$- (1 - u_1) C_1^V - u_1 k + \left\{ u_2 \left[ y\hat{\theta}_2 - \hat{\theta}_2^2 \right] - (1 - u_2) V_2 - u_2 k \right\}$$

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\[ \begin{align*}
U_1 &= \frac{1}{2} \dot{\theta}_1^2 + U_2 \\
V_1 &= C_1^V + sU_2 + (1 - s)V_2 \\
u_2 &= u_1(1 - \dot{\theta}_1) + s(1 - u_1) \\
\dot{\theta}_2 &= \sqrt{2U_2}
\end{align*} \]

Then the first promise-keeping constraint gives me
\[ \dot{\theta}_1 = \sqrt{2(U_1 - U_2)} \] (4)

which again coincides with equation (2) of the competitive equilibrium (with a general \( U_1 - U_2 \)).

The problem becomes
\[ P_1(V_1, U_1, u_1) = \max_{U_2, V_2} u_1 \left[ \theta_1 \left( y - \dot{\theta}_1 - (U_2 - V_2) \right) \right] \\
- (1 - u_1)(V_1 - sU_2 - (1 - s)V_2) - u_1 k \\
+ \left[ u_1(1 - \dot{\theta}_1) + s(1 - u_1) \right] \left[ y\dot{\theta}_2 - \dot{\theta}_2^2 - k + V_2 \right] - V_2
\]

subject to (3) and (4).

Notice that all the terms containing \( V_2 \) cancel out since by the law of motion of \( u_2 \)
\[ V_2 \left[ u_1\dot{\theta}_1 + (1 - u_1)(1 - s) - (1 - u_2) \right] = 0 \]

so the problem can be rewritten as
\[ P_1(V_1, U_1, u_1) = \max_{U_2, \dot{\theta}_1, \dot{\theta}_2} u_1 \left[ \theta_1 \left( y - \dot{\theta}_1 - U_2 \right) \right] \\
- (1 - u_1)(V_1 - sU_2) - u_1 k \\
- u_1 k + \left[ u_1(1 - \dot{\theta}_1) + s(1 - u_1) \right] \left[ \dot{\theta}_2 \left( y - \dot{\theta}_2 \right) - k \right]
\]

subject to (3) and (4).

The key mechanism that drives inefficiency comes from the fact that the social planner is taking into account the impact of \( U_2 \) on both \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \), while in the competitive equilibrium the decision of job creation at the second period, \( \dot{\theta}_2 \), which determines \( U_2 \), does not internalize the impact on job creation at time 1, \( \dot{\theta}_1 \).
condition of the social planner problem is

\[ u_1 \frac{d\hat{\theta}_1}{dU_2} \left( y - 2\hat{\theta}_1 - U_2 \right) - u_1 \hat{\theta}_1 + (1 - u_1) s + \\
+ \left[ u_1 (1 - \hat{\theta}_1) + s (1 - u_1) \right] \frac{d\hat{\theta}_2}{dU_2} \left( y - 2\hat{\theta}_2 \right) - u_1 \frac{d\hat{\theta}_1}{dU_2} \left[ \hat{\theta}_2 (y - \hat{\theta}_2) - k \right] \\
= 0 \]

where

\[ \frac{d\hat{\theta}_1}{dU_2} = - \left[ 2(U_1 - U_2) \right]^{-\frac{1}{2}} = -\hat{\theta}_1^{-1} \]

\[ \frac{d\hat{\theta}_2}{dU_2} = \left[ 2U_2 \right]^{-\frac{1}{2}} = \hat{\theta}_2^{-1} \]

At the CE, from the constraint binding in the second period it follows that

\[ y^* \hat{\theta}_2 = k. \]

Hence, a competitive equilibrium allocation can be a solution for the social planner problem only if it satisfies the first order condition (5) which becomes:

\[ -u_1 \hat{\theta}_1^{-1} \left( y - \hat{\theta}_1 - U_2 \right) + \left[ u_1 (1 - \hat{\theta}_1) + s (1 - u_1) \right] \hat{\theta}_2^{-1} \left( y - \hat{\theta}_2 \right) = 0, \]

that is when the economy starts with the \( u_1^* \) that exactly balance the positive effect of \( U_2 \) on job creation at time 2 and its negative effect on job creation at time 1. That is, the level \( u_1^* \) which solves

\[ u_1^* \left( \hat{\theta}_1^{CE} \right)^{-1} \left( y - \hat{\theta}_1^{CE} - U_2^{CE} \right) = \left[ u_1^* (1 - \hat{\theta}_1^{CE}) + s (1 - u_1^*) \right] \left( \hat{\theta}_2^{CE} \right)^{-1} \left( y - \hat{\theta}_2^{CE} \right). \]

This is a simple linear equation in \( u_1^* \) so I can find the unique value of \( u_1^* \) that satisfies this equation (just need to ensure the solution is between 0 and 1). When the economy starts from any level \( u_1 \neq u_1^* \), the competitive equilibrium allocation can be improved upon by the social planner, as shown with the next simple numerical example.

### 3 Numerical example

Parameter values: \( y = .9, k = .02, s = .1. \)
Figure 1 plots the function the social planner wants to maximize over the promised utility to unemployed workers, that is

\[
\Phi (U_2; u_1) = u_1 \left[ \sqrt{2 (U_1^{CE} - U_2)} \left( y - \sqrt{2 (U_1^{CE} - U_2)} - U_2 \right) \right] - (1 - u_1) (V_1^{CE} - s U_2) \\
- u_1 k + \left[ u_1 \left( 1 - \sqrt{2 (U_1^{CE} - U_2)} \right) + s (1 - u_1) \right] \left[ \sqrt{2 U_2} \left( y - \sqrt{2 U_2} \right) - k \right]
\]

for different values of \( u_1 \).

When \( u_1 = u_1^* \) the figure shows that in fact

\[
P_1 \left( V_1^{CE}, U_1^{CE}, u_1^* \right) = \max_{U_2} \Phi \left( U_2; u_1^* \right) = 0
\]

which means that the competitive equilibrium is constrained efficient. When \( u_1 \neq u_1^* \) then

\[
P_1 \left( V_1^{CE}, U_1^{CE}, u_1 \right) > 0
\]

and the optimal \( U_2 \) is above (below) \( U_2^{CE} \) if \( u_1 \) is below (above) \( u_1^* \). This shows that the direction of the inefficiency can go in both ways. In particular, increasing \( U_2 \) has two opposite effects on welfare: on one hand, it increases the welfare of workers who end
up being unemployed at the end of period 1 by increasing $\theta_2$, but on the other hand it makes workers who are matched in period 1 worse off by decreasing $\theta_1$, because when workers’ outside option is higher firms need to pay higher wages and so create less jobs. There exists a specific value $u^*_1$ such that these two effects are exactly balanced. When the initial unemployment rate is higher than that cut-off, then the cost of reducing job creation today dominates and the social planner would like to reduce $U_2$. When, instead, the initial unemployment rate is lower than that cut-off, then benefits from increasing job creation tomorrow dominates and the social planner would like to increase $U_2$, as figure 1 shows.