Identification and Inference of Jumps using Hedge Fund Returns

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Abstract

Correlated jumps in returns on international equities are infrequent. Therefore, estimating the characteristics of these jumps from time series observations of equity returns is impractical. This paper uses portfolio returns on international equities to generate additional information about the jumps’ characteristics. These portfolio returns are obtained from a group of hedge funds who invest in international equities. Via a model of optimal demand, I exploit the link between the demand of hedge funds for international equities and the characteristics of jumps. On average, hedge funds expect a jump in international equity returns of 15% or larger every 4 years. In contrast, time series observations of international equity returns imply that one of these jumps is expected every 8 years.

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1. Introduction

Asset returns are characterized by jumps. These jumps tend to be correlated across many assets. From the risk management perspective, correlated jumps are important because they can reduce the benefits of investors from diversification. The traditional approach to estimate the probability distribution of jumps in asset returns is to take time series observations of asset returns to estimate a model of asset returns that can capture jumps.

A problem of using asset returns alone is possible underestimation of the risk of jumps in asset returns because historical data on asset returns contains too few realized large jumps. To overcome this problem, I use portfolio returns on international equities to generate additional information about the characteristics of jumps in returns on international equities. These portfolio returns are obtained from a group of hedge funds who invest in international equities.

I assume that these hedge funds are price takers, know the dynamics of asset returns and optimally invest in international equities. Thus, the demand of these hedge funds for international equities depends on the characteristics of jumps. Therefore, the portfolio returns of these hedge funds reflect the demand of hedge funds for international equities. In this paper, I exploit the link between the demand of hedge funds for assets and the characteristics of jumps to generate additional information about jumps.

To do this, I use two ingredients. First, I use a model of asset returns that can capture correlated jumps. I model security returns as Lévy processes, in which jumps
across assets are simultaneous but the jump size is allowed to differ across assets. Second, I use an economic model of optimal demand to find the link between the demand of hedge funds for assets and the dynamics of asset returns. I assume that hedge funds solve Merton’s consumption-portfolio selection problem, and I use the results in Aït-Sahalia, Cacho-Diaz, and Hurd (2006) to derive the optimal investment rule in closed-form.

Since the optimal portfolio weights are derived in closed-form, the Lévy-Khintchine representation for Lévy processes also allows me to obtain moments of the optimal portfolio return and asset returns in closed-form. In order to take advantage of the closed-form expressions, I propose a generalized method of moments (GMM) approach to estimate the probability distribution of jumps. Moments of returns in closed-form make the GMM technique simple to directly implement. This estimation approach permits me to exploit the link between optimal portfolio weights and the probability distribution of asset returns.

I provide Monte Carlo evidence to show that identification of the jump parameters and efficiency of their estimators significantly improves when using optimal portfolio and asset returns instead of using asset returns alone. I perform different Monte Carlo scenarios including a misspecification case.

Next, I estimate the probability distribution of simultaneous jumps in returns on international equities. I use international equity indices as the underlying assets. I use returns of a group of hedge fund who invest in international equities to construct a proxy for optimal portfolio returns. Hedge fund returns serve as a proxy for optimal portfolio returns on international equities. Hedge funds are among the most sophisti-
cated investors, probably closer to the ideal of rational investors than any other class of investors.

The findings of this paper include (i) finite sample evidence suggests that jump parameters estimated from both asset and optimal portfolio returns are unbiased and significantly more efficient than those estimated from asset returns alone. (ii) Using the $\chi^2$ measure, I test the goodness-of-fit of the economic model of optimal demand and do not reject it. (iii) On average, hedge funds are long on international jump risk. However, the exposure of hedge funds to international jumps is smaller than the estimated value of the exposure to jumps implied by international equities alone. (iv) On average, hedge funds expect a jump in returns on international equity of 15% or larger every 4 years. In contrast, time series observations of international equity returns imply that one of these jumps is expected every 8 years. I propose a peso-problem interpretation of this result: hedge funds account for the risk of simultaneous market crashes, but actual realizations of these jumps are too infrequent to be consistently reflected in estimates drawn from the time series of international returns alone.

This paper relates to the literature that estimates financial models incorporating jump processes. Related papers include Schaumburg (2001) and Yu (2007) using maximum likelihood, Eraker, Johannes, and Polson (2003) based on Markov chain Monte Carlo, Chernov, Gallant, Ghysels, and Tauchen (2003) using the efficient method of moments, Das and Uppal (2004) based on the method of moments, and references therein. The econometric techniques in these papers employ asset returns data only. In contrast, the empirical results presented in this paper are based on estimates ob-
tained from both portfolio and asset returns. The one-to-one correspondence from portfolio returns, in conjunction with asset returns, to portfolio weights allows parameters governing the distribution of asset returns to be more accurately estimated from both asset and portfolio returns than from asset returns alone.

Papers similar in spirit to this work are Aït-Sahalia, Wang, and Yared (2001), Eraker (2004), Chernov and Ghysels (2000) and Pan (2002) who estimate the distribution of returns from joint asset returns and options data. Whereas the previous literature has focused on estimating asset-specific jumps, this paper studies correlated jumps across many assets.

This paper also contributes to the growing literature on hedge funds. Fung and Hsieh (1997), Ackermann, McEnally, and Ravenscraft (1999), Agarwal and Naik (2000), and Brown and Goetzmann (2003) investigate properties of hedge fund returns. Brown, Goetzmann, and Park (2000) and Fung and Hsieh (2000) study the impact of hedge funds over the Asian crisis and other market events. Also, Brunnermeier and Nagel (2004) analyzes the position of hedge funds in technology stocks during the technology bubble. In contrast to these empirical studies, my motivation is to use hedge fund returns to construct a proxy for optimal portfolio returns to estimate the distribution of jumps in returns on international equity markets.

The rest of the paper is organized as follows. In section 2, I give different types of arguments explaining why a model of optimal asset holdings is needed to estimate jump parameters. Next, in Section 3, I briefly summarize the model of Aït-Sahalia, Cacho-Diaz, and Hurd (2006) with the objective of obtaining the investor's optimal demand for risky assets in the presence of jumps. Section 4 analyses the optimal
portfolio return and defines a measure for the exposure of the optimal portfolio to jumps. In Section 5, I present the estimation approach. I present in Section 6 Monte Carlo evidence to show that indeed parameters of the underlying data generating process can be recovered from joint observations on asset and optimal portfolio returns and that these estimates are significantly better than the estimates from asset returns alone. Data are described in Section 7. Finally, I present in Section 8 the empirical results. I conclude in Section 9.
2. Why a model of optimal asset holdings is needed

2.1. Difficulty in identifying jump parameters

Before presenting the model to obtain the link between the investors’ demand and the jump parameters, I describe intuitively in this section why identification of jump parameters from asset prices alone could be expected to be difficult.

Consider for now the jump-diffusion specification

$$\frac{dS_t}{S_{t-}} = \mu dt + \sigma dW_t + J_t dN_t$$

(2.1)

where $S_t$ denotes the price of an asset. $W_t$ denotes a standard Brownian motion and $N_t$ a Poisson process with arrival rate $\lambda$. The log-jump size $\log(J_t)$ is a Gaussian random variable with mean $\xi$ and variance $\varphi^2$.

Identification, in the usual sense, is that the distribution of the data at the true parameters is different than that at any other possible parameter values. For any jump-diffusion process, there is positive probability that no jump occurs in a given finite time interval. When no jump occurs, the jump parameters $(\lambda, \xi, \varphi)$ cannot be identified by definition.

Figure 1 plots two paths using the same jump-diffusion process for $S_t$. One of the paths does not have jumps even though the process is allowed to jump with positive probability, $\lambda > 0$. 
Figure 1: Prices and log-returns simulated from a jump-diffusion model with $\lambda = 1$. These plots show a path with no jumps (left side) and a path with jumps (right side). The left plot has no jumps even though $\lambda > 0$.

2.2. Too few jumps

Now consider the right plots in Figure 1. In these paths, the asset price and log-return clearly exhibit jumps. However, the difficulty in estimating the jump parameters is that jumps are too infrequent. It cannot be expected to accurately estimate both $\lambda$ and the jump-size parameters $(\xi, \varphi)$ from observations of asset prices in a finite time interval.
2.3. Disentangle jumps from diffusions

The final intuition for the difficulty in identifying jumps lies in the effect of time aggregation, which in the present case takes the form of time smoothing. Just like a moving average is smoother than the original series, prices or returns observed over long periods are smoother than those observed over shorter horizons. In particular, jumps get averaged out. In discretely sample data, disentangle jumps from diffusions becomes difficult, see Aït-Sahalia (2004) for more details on this issue. This problem takes place even in the case where the jump process exhibits an infinite number of small jumps in any finite time interval, such as infinite activity Lévy jumps.

2.4. Finite sample estimators

A legitimate question at this point is how bad finite sample estimators from observations of asset returns alone are. I report the results of Monte Carlo simulations designed to examine the behavior of finite sample estimators when asset prices are observed once a day.

Starting with the jump-diffusion in equation (2.1), I simulate 5000 sample paths, each containing 1000 daily observations ($\Delta = 1/252$). In particular, $\log(J_t)$ is a normal random variable with mean $\xi = -0.1$ and standard deviation $\varphi = 0.03$. On average the asset jumps once a year $\lambda = 1$. These jump parameters are realistic based on actual estimates for stock index returns. I set the value of $\sigma$ at a realistic value, $\sigma = 0.3$. For simplicity, I assume that $\mu = 0$ and $\varphi$ are known.

I choose the parameters $(\sigma, \xi, \lambda)$ to minimize the squared deviation of the first four
Table 1: Parameter estimates from simulations.

<table>
<thead>
<tr>
<th></th>
<th>$\sigma$</th>
<th>$\xi$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>True Value</td>
<td>0.30</td>
<td>-0.10</td>
<td>1.00</td>
</tr>
<tr>
<td>Mean Estimate</td>
<td>0.30</td>
<td>-0.10</td>
<td>2.23</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.01</td>
<td>0.07</td>
<td>2.27</td>
</tr>
</tbody>
</table>

This table summarizes the results of 5000 Monte Carlo simulations. With each simulation, 1000 days of log-returns are observed. The parameters are estimated by minimizing the square of the difference between the first four unconditional moments of $\log(S_{t+\Delta}/S_t)$ and the moments implied by the simulated data.

The histograms show that the estimates of $(\xi, \lambda)$ are spread out on a wide interval. The estimates are biased and their sample variances are large. These results show that jump parameter estimates from asset prices alone on finite time intervals provide a bad approximation to the true jump parameter values. This analysis provides arguments for including additional identification information other than the asset returns.

2.5. What a model adds

Because jump parameters cannot be accurately estimated from observations of asset returns in a small time interval, I propose to use portfolio returns on international
Figure 2: Small sample distributions of the estimators of \((\xi, \lambda)\) by minimizing the square of the difference between the first four unconditional moments of \(\log(S_{t+\Delta}/S_t)\) and the moments implied by the simulated data.

equities to generate additional identification information. These portfolio returns are obtained from a group of hedge funds who invest in international equities. This group of hedge funds knows the dynamics of asset returns and, therefore, the probability distribution of jumps in returns.

Given the dynamics of asset returns, hedge funds optimally choose how much of their wealth to allocate in the risky assets. Thus, the demand of hedge funds for these assets must depend on the characteristics of the probability distribution of jumps.

To uncover the link between the demand of hedge funds for assets and the jumps’ characteristics, I will need a model relating the dynamics of asset returns to the optimal demand of hedge funds for assets.
3. Model of hedge funds’ optimal demand

In this section, I briefly present the economic model of Aït-Sahalia, Cacho-Diaz, and Hurd (2006) with the objective of obtaining the optimal demand of hedge funds for risky assets in the presence of jumps.

This theoretical model serves as an ingredient for the empirical analysis that I will conduct below.

3.1. Model of international equity returns

I assume that there are $m$ regions of the world, each containing $k$ countries. That is, a hedge fund selects the amounts to be held in $n = mk$ international equities and a riskless asset at times $t \in [0, \infty)$. The available investment opportunities consist of a riskless asset with price $S_{0,t}$ and $n$ risky assets, one per country, with prices $S_t = [S_{1,t}, \ldots, S_{n,t}]'$. Asset prices follow the exponential Lévy dynamics

$$
\frac{dS_{i,t}}{S_{i,t-}} = (r + R_i) dt + \sum_{j=1}^{n} \sigma_{i,j} dW_{j,t} + J_t dY_t, \quad i = 1, \ldots, n \quad (3.1)
$$

$$
\frac{dS_{0,t}}{S_{0,t}} = r dt, \quad (3.2)
$$

with a constant rate of interest $r \geq 0$. $W_t = [W_{1,t}, \ldots, W_{n,t}]'$ is an $n$–dimensional standard Brownian motion. $Y_t$ is a Lévy pure jump process with Lévy measure $\nu(dz) = \lambda dz/z$ and $z \in [-1, 0]$. This measure $\nu$ satisfies $\int_{R/[-a,a]} \nu(dz) < \infty$ for any $a > 0$, and $\int_{\mathbb{R}} \min(1, z^2) \nu(dz) < \infty$, $l = 1, \ldots, m$. So jumps across assets are simultaneous but the jump size is allowed to differ across assets.
The Lévy pure jump processes and the Brownian motions are mutually independent and the jumps have support on \((-1, \infty)\) to guarantee the positivity (or limited liability) of \(S_i\). The constant \(J_i\) is asset \(i\)'s scaling jump factor. The quantities \(R_i\), and \(\sigma_{ij}\) are constant parameters. I write \(\mathbf{R} = [R_1, \ldots, R_n]'\), \(\mathbf{J} = [J_1, \ldots, J_n]'\) and \(\Sigma = \mathbf{\sigma}\mathbf{\sigma}'\) where
\[
\mathbf{\sigma} = \begin{pmatrix}
\sigma_{1,1} & \cdots & \sigma_{1,n} \\
\vdots & \ddots & \vdots \\
\sigma_{n,1} & \cdots & \sigma_{n,n}
\end{pmatrix}.
\] (3.3)
I assume that \(\Sigma\) is a nonsingular matrix.

### 3.1.1. Region jumps

Let me define the vector \(\mathbf{1}_l\) as the \(n\)–vector with ones placed in the rows corresponding to the \(l\)–block and zeros everywhere else:
\[
\mathbf{1}_l = [0, \ldots, 0, 1_{\text{Region } l}, \ldots, 1, 0, \ldots, 0]',
\] (3.4)
where the first 1 is located in the \(k (l - 1) + 1\) coordinate, and the last one in the \(kl + 1\) coordinate.

To capture the notion of regions, I assume that each jump vector is of the form
\[
\mathbf{J} = \sum_{l=1}^{m} j_l \mathbf{1}_l = [j_{1,1}, \ldots, j_{1,n}, j_{2,1}, \ldots, j_{2,n}, \ldots, j_{m,1}, \ldots, j_{m,n}]',
\] (3.5)
meaning that countries within a given region have the same response to the arrival of a jump, i.e. to a \(dY_t\), but the proportional response \(j_l\) of firms of different regions to the arrival of a jump can be different. So the jump vector is unrestricted linear combinations of the \(m\) vectors \(\{\mathbf{1}_l\}_{l=1, \ldots, m}\).
3.1.2. Multifactor Brownian Risk

I assume a block-structure for the variance-covariance matrix of returns to capture the notion of regions,

\[
\Sigma_{m \times m} = \begin{pmatrix}
\Sigma_{1,1} & \Sigma_{1,2} & \cdots \\
\Sigma_{2,1} & \ddots & \Sigma_{2,m} \\
\vdots & \ddots & \Sigma_{m-1,m} \\
\end{pmatrix}
\]  

(3.6)

with within-region blocks

\[
\Sigma_{l,l}^{k \times k} = v_l^2 \begin{pmatrix}
1 & \rho_{l,l} & \cdots \\
\rho_{l,l} & \ddots & \rho_{l,l} \\
\vdots & \ddots & 1 \\
\end{pmatrix},
\]

(3.7)

and across-region blocks

\[
\Sigma_{l,s}^{k \times k} = v_l v_s \begin{pmatrix}
\rho_{l,s} & \rho_{l,s} & \cdots \\
\rho_{l,s} & \ddots & \rho_{l,s} \\
\vdots & \ddots & \rho_{l,s} \\
\end{pmatrix},
\]

(3.8)

where the correlation between assets in region \( l \) and in region \( s \) have the same correlation, where \( 1 > \rho_{l,l} > \rho_{l,s} \) and \( \rho_{l,l} \geq -1/(k - 1) \) and \( \rho_{l,s} \geq -1/(n - 1) \). This factor structure means that assets in a given region have the same variance and correlation, but assets in different regions have different variances and correlations.

In an asset pricing framework, this corresponds to a multifactor model for the returns process with \( m \) common Brownian factors and \( n \) idiosyncratic Brownian shocks.
3.2. Hedge funds’ optimal portfolio

Let \( \omega_{0,t} \) denote the percentage of wealth (or portfolio weight) invested by a hedge fund at time \( t \) in the riskless asset and \( \omega_t = [\omega_{1,t}, \ldots, \omega_{n,t}]' \) denote the vector of portfolio weights in each of the \( n \) assets. The portfolio weights satisfy

\[
\omega_{0,t} + \sum_{i=1}^{n} \omega_{i,t} = 1. \tag{3.9}
\]

I assume that the investor consumes \( C_t \) at time \( t \). In the absence of any income derived outside his investments in these assets, the investor’s wealth, starting with the initial endowment \( X_0 \), follows the dynamics

\[
dX_t = -C_t dt + \omega_{0,t} X_t \frac{dS_{0,t}}{S_{0,t}} + \sum_{i=1}^{n} \omega_{i,t} X_t \frac{dS_{i,t}}{S_{i,t}}
= (r X_t + \omega_t' R X_t - C_t) dt + X_t \omega_t' \sigma dW_t + X_t \sum_{l=1}^{m} \omega_l J_l dY_{l,t}. \tag{3.10}
\]

Given that the hedge fund knows the model parameters, the hedge fund’s problem at time \( t \) is then to pick the consumption and portfolio weights \( \{C_s, \omega_s\}_{t \leq s \leq \infty} \) which maximize the infinite horizon, discounted at rate \( \beta \), expected utility of consumption.

\[
V(X_t, t) = \max_{\{C_s, \omega_s; t \leq s \leq \infty\}} E_t \left[ \int_t^\infty e^{-\beta s} U(C_s) \, ds \right] \tag{3.11}
\]
subject to the dynamics of his discounted wealth (3.10), and with \( X_t \) given.

In order to determine the optimal portfolio weights, I consider investors with power utility, \( U(c) = c^{1-\gamma} / (1 - \gamma) \) for \( c > 0 \) and \( U(c) = -\infty \) for \( c \leq 0 \) with CRRA coefficient \( \gamma \). Using stochastic dynamic programming and the appropriate form of Itô’s lemma for semi-martingale processes, the Hamilton-Jacobi-Bellman equation...
characterizing the optimal solution to the investor’s problem reduces to:

\[ \omega_t^* = \arg \min_{\{\omega_t\}} f(\omega_t). \]  
\( (3.12) \)

where the functions

\[ f(\omega) = -\omega'R + \frac{\gamma}{2} \omega'^{\prime} \Sigma \omega + \psi(\omega'J) \]  
\( (3.13) \)

and

\[ \psi(\omega'J) = -\frac{1}{(1-\gamma)} \int_{-1}^{0} \left( (1 + \omega'Jz)^{1-\gamma} - 1 \right) \frac{dz}{z}. \]  
\( (3.14) \)

are both convex.

As in Aït-Sahalia, Cacho-Diaz, and Hurd (2006), I look for a vector of optimal portfolio weights in the form

\[ \omega = \sum_{l=1}^{m} \frac{\bar{\omega}_l}{k} 1_l + \omega^\perp = \bar{\omega} + \omega^\perp. \]  
\( (3.15) \)

where \( \bar{\omega}_l \) is a scalar and \( \omega^\perp \) is a \( n \)-vector orthogonal to the \( k \)-vector of ones, \( 1_l \). I decompose the vector of expected excess returns on the same basis as above,

\[ R = \sum_{l=1}^{m} \tilde{r}_l 1_l + R^\perp = \bar{R} + R^\perp \]  
\( (3.16) \)

where \( R^\perp \) is orthogonal to each \( 1_l \) and has the form

\[ R^\perp = [R_{1}^\perp, \ldots, R_{m}^\perp]' . \]  
\( (3.17) \)

In order to obtain a closed-form solution, I consider investors with CRRA coefficient \( \gamma = 2 \). Equations (3.12)-(3.13) lead to the optimal solution \( \omega^{\perp*} = [\omega_1^{\perp*}, \ldots, \omega_m^{\perp*}]' \) with

\[ \omega_l^{\perp*} = \frac{1}{2v^2_l (1 - \rho_{l,i})} R_{l}^\perp \]  
\( (3.18) \)
for \( l = 1, \ldots, m \).

For the optimal solution \( \bar{\omega}^* = [\bar{\omega}_1^*, \ldots, \bar{\omega}_m^*]' \), let \( j = [j_1, \ldots, j_m]' \), \( \bar{r} = [\bar{r}_1, \ldots, \bar{r}_m]' \),
\[ A = \frac{k}{2} j' K^{-1} \bar{r}, \quad B = \frac{k}{2} j' K^{-1} j, \]
and
\[
K = \begin{pmatrix}
v_1^2 \left( 1 + (k - 1) \rho_{1,1} \right) & \cdots & kv_1 v_m \rho_{1,m} \\
\vdots & \ddots & \vdots \\
k v_m v_1 \rho_{m,1} & \cdots & v_m^2 \left( 1 + (k - 1) \rho_{m,m} \right)
\end{pmatrix},
\tag{3.19}
\]
then
\[
\bar{\omega}^* = \frac{k}{2} K^{-1} \bar{r} + \frac{k(y - A)}{2B} K^{-1} j
\tag{3.20}
\]
where
\[
y = \frac{1 + A - \sqrt{(1 - A)^2 + 4\lambda B}}{2}.
\tag{3.21}
\]

The component \( \omega_{l}^* \) of the optimal portfolio doesn’t depend on the jump parameters, \( \lambda \) and \( j \). However, the component \( \bar{\omega}^* \) is a function of the jump parameters. In order to identify the jump parameters, one would like to exploit the link between the optimal portfolio weights, \( \omega^* \), and the jump parameters.

Unfortunately, data on investors’ asset allocation are scarce. Nevertheless, the investors’ demand can be identified from the investors’ portfolio returns in conjunction with asset returns. In the next section, I show how the component \( \bar{\omega}^* \) can be identified from observations of asset and portfolio returns.
4. Exposure of the optimal portfolio to jumps

This section analyses the investor’s portfolio return and defines a measure for the exposure of the investor’s portfolio to jumps.

Because the investor’s portfolio can be expressed as

$$\omega = \bar{\omega} + \omega^\perp,$$

then, the instantaneous excess return, $dP_t$, of this portfolio,

$$dP_t = \sum_{i=1}^{n} \omega_i \left( \frac{dS_i}{S_i} - rd_{t} \right),$$

can be decompose into two parts

$$dP_t = d\bar{P}_t + dP_t^\perp$$

where the component

$$dP_t^\perp = \omega^{\perp'} R dt + \omega^{\perp'} \sigma dW_t$$

does not have exposure to jumps, and the return

$$d\bar{P}_t = \sum_{l=1}^{m} \bar{\omega}_l \bar{r}_l dt + \sum_{l=1}^{m} \frac{\bar{\omega}_l}{k} \bar{y}_l \sigma dW_t + \sum_{l=1}^{m} \bar{\omega}_l j_l dY_t$$

has exposure to jumps.

Hence, the log-return $d\bar{P}_t$ is a linear combination of excess returns of equally weighted portfolios within regions, that is

$$d\bar{P}_t = \sum_{l=1}^{m} \bar{\omega}_l \left[ \bar{r}_l dt + \frac{1}{k} \bar{y}_l \sigma dW_t + j_l dY_t \right]$$

$$= \sum_{l=1}^{m} \bar{\omega}_l \frac{1}{k} \sum_{i \text{ in region } l} \left( \frac{dS_{i,t}}{S_{i,t}} - rd_{t} \right)_{dE_l}$$
where $dE_l$ is the excess return of the equally weighted portfolio of assets on region $l$. Here we can notice that the correlation between the two components of the portfolio returns, $d\tilde{P}_t$ and $dP^\perp_t$, is zero,

$$\text{cov}(d\tilde{P}_t, dP^\perp_t) = \sum_{l=1}^{m} \tilde{\omega}_l \frac{1}{k} \sum_l \omega^l \Sigma \omega^l = 0 \quad (4.7)$$

In a nutshell, we can represent the returns of the optimal portfolio as the sum of two uncorrelated components where only one of the parts has exposure to jumps. This component, $d\tilde{P}_t$, is a linear combination of regional equally weighted portfolio returns, $dE_l$. This decomposition will allow me to identify the part of the optimal portfolio that depends on jumps, $\tilde{\omega}$.

Notice the jump term in (4.5) can be expressed as

$$\sum_{l=1}^{m} \tilde{\omega}_l j_l dY_t = ydY_t \quad (4.8)$$

by definition of $\tilde{\omega}$. Thus, I define $y$ as the exposure of the optimal portfolio to jumps. The larger the $y$ is, the stronger the exposure of the portfolio to jumps is.

### 5. Estimation

In this section, I focus on how to estimate the parameters of the parametric model of asset returns using joint time-series data $\{P_t, E_t\}$ on portfolio and asset log-prices. Treating the risk aversion coefficient $\gamma$ and the Lévy measure $\nu$ as given, my focus in this section is on the estimation of the jump parameters $\{\lambda, J\}$.

Given that the density of returns is not known in closed-form, the joint density function of the observables $P_t$ and $E_t$ is complicated to estimate by maximum like-
likelihood techniques. In order to take advantage of the closed-form expressions of the optimal portfolio weights and of the moments of returns, I use the generalized method of moments (GMM) technique developed by Hansen (1982). In the remainder of this section, I provide a detail description of the estimation procedure and the GMM estimators.

5.1. GMM estimators

Fixing some time interval $\Delta$, assume there is a portfolio return, $r^*_t$, with dynamics given by

$$r^*_t = P_t - P_{t-\Delta} + \eta_t$$

(5.1)

where $\eta$ is a white noise process uncorrelated with the log-value of the optimal portfolio, $P$. I sample the continuous-time process $\{r^*_t, E_t\}$ at discrete times $\{0, \Delta, 2\Delta, \ldots, T\Delta\}$ and denote the sampled process $\{r^*_{u\Delta}, E_{u\Delta}\}$ by $\{r^*_u, E_u\}$.

Let

$$r_{l,u} = E_{l,u} - E_{l,u-1} \text{ for } l = 1, \ldots, m$$

(5.2)

denote the date-$u$ excess returns, $r_u = [r_{1,u}, \ldots, r_{m,u}]'$, and

$$\varepsilon_u = P_u^\perp - P_{u-1}^\perp - E[P_u^\perp - P_{u-1}^\perp] + \eta_u.$$  

(5.3)

It is easy to see from the definition of $P$ and from (4.7) that

$$r^*_u = \alpha + \omega'_n r_u + \varepsilon_u$$

(5.4)

and

$$E[\varepsilon_u] = 0 \text{ and } E[\varepsilon_u \otimes r_u] = 0$$

(5.5)
with \( \alpha = E \left[ P_u^1 - P_{u-1}^1 \right] . \)

Taking advantage of the Lévy-Khintchine representation for Lévy processes (see, e.g., Sato (1999), for further details), I derive the unconditional moments for the asset returns, \( r_u \). The Lévy-Khintchine formula of \( r_u \) is

\[
E \left[ e^{i s r_u} \right] = e^{\Delta \Psi(s)}, \quad \text{for } s \in \mathbb{R}^m
\]

(5.6)

with

\[
\Psi(s) = i r' s - \frac{1}{2} s' K s + \int \left( e^{i s' \mathbf{z}} - 1 \right) \nu(dz)
\]

(5.7)

Then, the moments of \( r_u \) are the following: for \( s \geq l = \{1, \ldots, m\} ,

\[
E[r_{l,u}] = \Delta (\bar{r}_l - j_l \lambda)
\]

\[
E[(r_u - E[r_u]) (r_u - E[r_u])'] = \Delta \left( \frac{1}{k} K + \frac{1}{2} j j' \lambda \right)
\]

\[
E[(r_{l,u} - E[r_{l,u})] (r_{s,u} - E[r_{s,u})] = -\frac{\Delta}{3} j_l^2 j_s \lambda
\]

(5.8)

\[
E[(r_{l,u} - E[r_{l,u})]^4 = \frac{\Delta}{4} j_l^4 \lambda + \frac{3}{4} \Delta^2 \left( j_l^2 \lambda + \frac{2}{k} \{K\}_{l,t} \right)^2
\]

For notational convenience, I summarize the parameters to be estimated by \( \theta = \{\alpha, K, j, r, \lambda\} \). For \( s \geq l = \{1, \ldots, m\} , \) let the vector \( f_u(\theta) \) be

\[
f_u(\theta) = \begin{bmatrix}
\varepsilon_u \\
\varepsilon_u r_u \\
r_u - \Delta (\bar{r}_l - j_l \lambda) \\
(r_{l,t} - E[r_{l,t}]) (r_{s,t} - E[r_{s,t}]) - \Delta \left( \frac{\{K\}_{l,t} \lambda}{k} + \frac{1}{2} j m j_s \lambda \right) \\
(r_{l,u} - E[r_{l,u})] (r_{s,u} - E[r_{s,u})] - \frac{\Delta}{3} j_l^2 j_s \lambda \\
(r_{l,u} - E[r_{l,u})]^4 - \frac{\Delta}{4} j_l^4 \lambda - \frac{3}{4} \Delta^2 \left( j_l^2 \lambda + \frac{2}{k} \{K\}_{l,t} \right)^2
\end{bmatrix}
\]

20
Thus, we have a total of $2m + m(m + 1)/2 + 1$ parameters to match $3m + 2m(m + 1)/2 + 1$ moment conditions. Under the null hypothesis that the restrictions implied by (5.5) and (5.8) are true, $E[f_t(\theta)] = 0$. The GMM procedure consists of replacing $E[f_t(\theta)]$ with its sample counterpart, $g_T(\theta)$, using the $T$ observations where

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} f_t(\theta)$$

and then choosing parameter estimates that minimize the quadratic form,

$$J_T(\theta) = Tg_T'(\theta)W_Tg_T(\theta)$$

(5.10)

where $W_T$ is a positive-definite symmetric weighting matrix. Matrix differentiation shows that minimizing $J_T(\theta)$ with respect to $\theta$ is equivalent to solving the homogeneous system of equations (orthogonality conditions),

$$D'(\theta)W_Tg_T(\theta) = 0,$$

(5.11)

where $D(\theta)$ is the Jacobian matrix of $g_T(\theta)$ with respect to $\theta$.

Hansen (1982) shows that choosing $W_T = S^{-1}(\theta)$, where

$$S(\theta) = E[f_t(\theta)f_t'(\theta)],$$

(5.12)

results in the GMM estimator of $\theta$ with the smallest asymptotic covariance matrix. Designating an estimator of this covariance matrix as $\hat{S}(\theta)$, the asymptotic covariance matrix for the GMM estimate of $\theta$ is

$$\frac{1}{T} \left( \hat{D}'(\theta) \hat{S}^{-1}(\theta) \hat{D}(\theta) \right)^{-1},$$

(5.13)
where $\dot{D}(\theta)$ is the Jacobian evaluated at the estimated parameters. This covariance matrix is used to test the significance of the individual parameters.

The minimized value of the quadratic form in (5.10) is distributed $\chi^2$ under the null hypothesis that the model is true with degrees of freedom equal to the number of orthogonality conditions net of the number of parameters to be estimated. This $\chi^2$ measure provides a goodness-of-fit test for the model. A high value of this statistic means that the model is misspecified.

6. Monte Carlo analysis

In this subsection, I present Monte Carlo evidence that indeed population parameters of the underlying data generating process can be recovered from joint observations on asset and portfolio returns and that these estimates are significantly better than the estimates from asset returns alone.

Starting with the dynamics of assets, $S$, I simulate 5000 sample paths, each containing 125 observations. The length of each series coincides with observable monthly data on which the empirical results are based on. The population parameters of the dynamics of $S$ are chosen such that they have the same magnitude as the estimated parameters from the empirical results. In particular, for this Monte Carlo exercise the number of regions is $m = 4$, the number of countries per region is $k = 4$, and the jump parameter values are $(j_1, j_2, j_3, j_4, \lambda) = (0.42, 0.14, 0.13, 0.15, 0.74)$.

Once the values of $S$ are simulated, then I compute the optimal portfolio weights to simulate the optimal portfolio returns. I summarize the small sample behavior of
Table 2: Jump parameter estimates from simulations.

<table>
<thead>
<tr>
<th></th>
<th>$j_1$</th>
<th>$j_2$</th>
<th>$j_3$</th>
<th>$j_4$</th>
<th>$\lambda$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>True Value</td>
<td>0.42</td>
<td>0.14</td>
<td>0.13</td>
<td>0.15</td>
<td>0.74</td>
<td>0.11</td>
</tr>
<tr>
<td>Model I</td>
<td>0.39</td>
<td>0.14</td>
<td>0.12</td>
<td>0.14</td>
<td>0.78</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.05)</td>
<td>(0.06)</td>
<td>(0.05)</td>
<td>(0.16)</td>
<td>(0.03)</td>
</tr>
<tr>
<td>Model II</td>
<td>0.31</td>
<td>0.10</td>
<td>0.09</td>
<td>0.11</td>
<td>2.24</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>(0.15)</td>
<td>(0.06)</td>
<td>(0.07)</td>
<td>(0.06)</td>
<td>(4.68)</td>
<td>(0.24)</td>
</tr>
<tr>
<td>Model $II^*$</td>
<td>0.33</td>
<td>0.11</td>
<td>0.09</td>
<td>0.12</td>
<td>2.14</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>(0.15)</td>
<td>(0.06)</td>
<td>(0.07)</td>
<td>(0.06)</td>
<td>(5.29)</td>
<td>(0.23)</td>
</tr>
</tbody>
</table>

This table summarizes the results of 5000 Monte Carlo simulations. Mean estimates and standard deviation (in parenthesis) are reported. With each simulation 125 months of returns are observed. For Model I the parameters are estimated from joint asset and portfolio returns, using moments in (5.8) and (5.5). For Model II the parameters are estimated from asset returns alone, using moments in (5.8). For Model $II^*$ the parameters are estimated from asset returns alone, using moments in (6.1) in addition to (5.8).

As a result, on average the jump parameter estimates of Model I are closer to the true parameters than the estimates of Model II. One can also see that on average the estimated value of $\lambda$ from model II is clearly overestimated and the jump scaling parameters, $j$, are underestimated. In fact, the standard deviations across the Monte Carlo simulations of Model I are low compare to their parameter values.

This Monte Carlo exercise shows that jump estimates from joint asset and portfolio returns are not only significantly more efficient than estimates drawn from asset
returns alone but also extremely close to the true parameters.

At first glance, the result that Model I yields much better estimates than Model II is surprising since both models have the same source of randomness. One may wonder whether the improvement of the estimates is because Model I uses more moments than Model II does.

To address this issue, I estimate the jump parameters with Model II together with the additional moments

$$E[(r_{l,u} - E[r_{l,u}])^2 (r_{s,u} - E[r_{s,u}])] = -\frac{\Delta}{3} j_l^2 j_s \lambda \text{ for } s < l = \{1, ..., m\}$$

(Model II*) but still without using the moment conditions given by the portfolio returns. The parameter estimates for Model II* are reported in Table 2. These parameter estimates do not suggest that the addition of arbitrary moments of asset returns alone is the cause for better parameter estimates.

The key is the additional identification information of the jump parameters given by the investor’s demand. Identification of the jump parameters is weak from asset returns alone. On the one hand, I am using monthly observations which makes difficult to disentangle tiny jumps from diffusions, see Aït-Sahalia (2004). On the other hand, for any finite time interval there is positive probability that some of the paths have no big jumps. Identification of the jump parameters, then, comes from the portfolio return through its portfolio weights.
6.1. Identification intuition

To illustrate the last point about identification, consider a one region case, \( m = 1 \), with one country, \( k = 1 \), and the risky asset follows a jump-diffusion process with jump intensity \( \lambda \). Suppose that an econometrician knows all parameter values but \( \lambda \).

The econometrician wants to estimate \( \lambda \) from observations on the asset excess return, \( X \), in a time interval where no jumps have occurred. It is clear that he will not be able to identify \( \lambda \) since those observations could have been generated by different values of \( \lambda \). On the other hand, suppose that the econometrician wants to estimate \( \lambda \) not only from the asset excess return but also from the optimal portfolio excess return, \( Y \).

He knows that the optimal portfolio weight, \( \omega(\lambda) \), on \( X \) depends on \( \lambda \). Also, notice that the portfolio weight \( \omega(\lambda) \) can be recovered from observations on \( X \) and \( Y \) since \( Y = \omega(\lambda)X \). If there is a one-to-one correspondence between \( \omega(\lambda) \) and \( \lambda \), then the jump intensity, \( \lambda \), is completely identified from the portfolio weight. This last result holds even if no jumps are observed in \( X \).

6.2. Robustness checks to misspecification

In the previous Monte Carlo exercise I simulated portfolio returns from investors with a CRRA coefficient of \( \gamma = 2 \). In this subsection, I analyze how robust are the estimates when the investors have a different CRRA coefficient from \( \gamma = 2 \) but still estimating the parameters assuming that the investors’ CRRA is \( \gamma = 2 \). Table 3 reports the parameter estimates. Surprisingly, the estimates of the jump scaling
Table 3: Estimates from simulations using different CRRA coefficients.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$j_1$</th>
<th>$j_2$</th>
<th>$j_3$</th>
<th>$j_4$</th>
<th>$\lambda$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.39</td>
<td>0.14</td>
<td>0.12</td>
<td>0.14</td>
<td>0.78</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.05)</td>
<td>(0.06)</td>
<td>(0.05)</td>
<td>(0.16)</td>
<td>(0.03)</td>
</tr>
<tr>
<td>3</td>
<td>0.40</td>
<td>0.14</td>
<td>0.12</td>
<td>0.14</td>
<td>0.81</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.05)</td>
<td>(0.06)</td>
<td>(0.05)</td>
<td>(0.16)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>4</td>
<td>0.40</td>
<td>0.14</td>
<td>0.12</td>
<td>0.15</td>
<td>0.82</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.05)</td>
<td>(0.06)</td>
<td>(0.05)</td>
<td>(0.16)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>5</td>
<td>0.40</td>
<td>0.14</td>
<td>0.12</td>
<td>0.14</td>
<td>0.84</td>
<td>0.04</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.05)</td>
<td>(0.06)</td>
<td>(0.05)</td>
<td>(0.16)</td>
<td>(0.01)</td>
</tr>
</tbody>
</table>

This table summarizes the results of 5000 Monte Carlo simulations. Mean estimates and standard deviation (in parenthesis) are reported. With each simulation 125 months of returns are observed. The portfolio returns are constructed using five different values of CRRA, $\gamma$. The parameters are estimated from joint asset and portfolio returns, using moments in (5.8) and (5.5).

parameters, $j$, are insensitive to the value of $\gamma$. However, the jump parameter $\lambda$ slightly increases as the investor’s risk aversion coefficient increases, $\gamma$. Overall, the jump parameter estimates are robust to possible misspecification on the investor’s risk aversion coefficient.

7. Data

The empirical analysis in this paper focus on estimating the probability distribution of simultaneous jumps in returns on international equities. I use international equity
indices as the underlying assets, $S$. The index data is from Global Financial Data. I construct indices using hedge fund returns to proxy the portfolio return $r^*$. Hedge fund returns will serve as proxy for the optimal portfolio return. Hedge fund data is from TASS database.\footnote{I thank Burton Malkiel for providing me with the hedge fund data.}

### 7.1. International indices

The data for the equity markets consists of the month-end U.S. dollar values of the total return equity indices for the period March 1994 to May 2004 for Argentina, Belgium, Brazil, Canada, Chile, China, France, Germany, Hong Kong, Japan, Mexico, Switzerland, Taiwan, United Kingdom, and United States. For the riskless asset I use the US three-moth Tbill rate.

I divide these countries in four economic regions, $m = 4$. Each economic region consists of four countries, $k = 4$. The CJUU region consists of Canada, Japan, United Kingdom and United States. The Latin-American region (LA) consists of Argentina, Brazil, Chile and Mexico. The European region (EU) consists of Belgium, France, Germany and Switzerland. The Asian region (AS) includes China, Hong Kong, Singapore and Taiwan.

Table 4 reports descriptive statistics for the average of the monthly, $\Delta = 1/12$, excess log-returns for country indices on Region $l$, $r_{l,u}$, which is defined by

$$r_{l,u} = \frac{1}{k} \sum_{i \text{ in region } l} \log \left( \frac{S_{i,u}}{S_{i,u-1}} \right) - r\Delta \quad (7.1)$$

where $S_{i,u}$ is the US$ value of the index of country $i$ at time $u$. We observe from Table
Table 4: Descriptive statistics for equity indices log-returns.

<table>
<thead>
<tr>
<th></th>
<th>CJUU</th>
<th>LA</th>
<th>EU</th>
<th>AS</th>
<th>Avg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0017</td>
<td>-0.0014</td>
<td>0.0031</td>
<td>-0.0064</td>
<td>-0.0008</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.0421</td>
<td>0.0882</td>
<td>0.0472</td>
<td>0.0773</td>
<td>0.0637</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.6526</td>
<td>-0.9927</td>
<td>-0.9285</td>
<td>0.1528</td>
<td>-0.6053</td>
</tr>
<tr>
<td>Excess kurtosis</td>
<td>3.6501</td>
<td>6.1261</td>
<td>5.3420</td>
<td>3.9205</td>
<td>4.7597</td>
</tr>
</tbody>
</table>

This table gives the first four moments of the average of the monthly excess log-returns in U.S. dollar terms for country indices within a region. The data for the country indices are from March 1994 to May 2004. Each economic region consists of four countries. The CJUU region consists of Canada, Japan, United Kingdom and United States. The Latin-American region (LA) consists of Argentina, Brazil, Chile and Mexico. The European region (EU) consists of Belgium, France, Germany and Switzerland. The Asian region (AS) includes China, Hong Kong, Singapore and Taiwan.

It is evident that the excess kurtosis of returns is substantially greater than that for Gaussian distributions. The excess kurtosis in the data ranges from 3.65 for the CJUU region to 6.13 for the Latin-American region. The average kurtosis for the regions is more than 4.75. Jumps in returns could be a cause for these kurtosis.

The correlations across regions range from 0.41 between the regions Europe and Asia, to 0.74 between the regions Europe and CJUU. The average correlation for the regions is more than 0.5. The correlations might come either from Brownians or from jumps. The most volatile region is Latin-American with a variance of 0.0078 and the least volatile region is CJUU with a variance of 0.0018. The average variance for the regions is more than 0.004.
Table 5: Correlations and covariances between regions.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>CJUU</td>
<td>0.0018</td>
<td>0.6692</td>
<td>0.7361</td>
<td>0.5986</td>
<td></td>
</tr>
<tr>
<td>LA</td>
<td>0.0025</td>
<td>0.0078</td>
<td>0.5200</td>
<td>0.5816</td>
<td></td>
</tr>
<tr>
<td>EU</td>
<td>0.0015</td>
<td>0.0022</td>
<td>0.0022</td>
<td>0.4113</td>
<td>0.5027</td>
</tr>
<tr>
<td>AS</td>
<td>0.0020</td>
<td>0.0040</td>
<td>0.0015</td>
<td>0.0060</td>
<td></td>
</tr>
</tbody>
</table>

This table gives the covariances (in italics) and the correlations of the monthly excess returns in U.S. dollar terms for the equally-weighted returns of indices within regions. The data for the country indices are from March 1994 to May 2004. Each economic region consists of four countries. The CJUU region consists of Canada, Japan, United Kingdom and United States. The Latin-American region (LA) consists of Argentina, Brazil, Chile and Mexico. The European region (EU) consists of Belgium, France, Germany and Switzerland. The Asian region (AS) includes China, Hong Kong, Singapore and Taiwan.

7.2. Hedge funds

The TASS database consists of monthly returns, assets under management and other fund-specific information for 4,781 individual funds from February 1977 to May 2004. The database is divided into two groups: "live" and "dead" funds. Hedge funds that are in the "live" group are considered to be active as of May 2004. Funds that are in the "dead" group have stopped reporting to TASS for liquidations or other unexplained issues. Out of the 4,781 funds, 2,900 funds are in the live group and 1,861 in the dead group.

In this article I employ hedge funds who invest in international equities. Fortunately, TASS indicates when a hedge fund is global and equity oriented.

I eliminate funds from the database that report only gross returns or that report
returns on quarterly -not monthly- basis. I also eliminate funds with less than nine years of reported returns. Finally, I drop funds that report estimated -not actual- returns leaving a final sample of ten live funds and twelve dead funds from March 1994 up to May 2004.

More than half of the hedge funds in this group are classified as long/short equity hedge. Their investment approach is fundamental and bottom up. And, their sector focus is large cap. The average correlation between hedge fund returns is more than 0.50.

With these funds, I construct an equally-weighted index. This index is the average of the monthly excess log-returns for hedge funds in U.S. dollar terms. I use this index as a proxy for the portfolio return, $r^*$. 

For robustness, I also consider an index constructed as the value-weighted average of the monthly excess log-returns for hedge funds in U.S. dollar terms.

Table 6 reports descriptive statistics for these two indices. We observe that on average hedge fund returns have higher skewness than those of international equity returns and have lower kurtosis than those of international equity returns. The statistics of hedge funds are consistent with the predictions of the model. Namely, hedging against correlated jumps generates returns with lower kurtosis and higher skewness.

Figure 3 plots the equally weighted hedge fund index together with the log-returns of the CJUU, Europe, Asian and Latin-American regions. The plot suggests that there are a few correlated large drops but no correlated large positive shocks.

For instance, some of the correlated large negative shocks in Figure 3 are the
Table 6: Descriptive Statistics for hedge fund indices.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Skewness</th>
<th>Excess kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equally-weighted funds</td>
<td>0.0014</td>
<td>0.0304</td>
<td>-0.1911</td>
<td>0.7972</td>
</tr>
<tr>
<td>Value-weighted funds</td>
<td>0.0047</td>
<td>0.0333</td>
<td>0.4510</td>
<td>3.3509</td>
</tr>
</tbody>
</table>

This table gives the first four moments of the equally-weighted and the value-weighted indices of the monthly excess log-returns in U.S. dollar terms for hedge funds. The data for the hedge funds are from March 1994 up to May 2004, and include observations of month-end values of ten live and twelve dead hedge funds. Only equity-oriented hedge funds with global focus are included.


One can immediately observe that the equally-weighted hedge fund index is less volatile than the international equity returns. Moreover, when there is a large correlated drop in equity returns, on average hedge fund returns don’t fall as much as equity returns do. This plot suggests that hedge funds recognize the presence of correlated jumps and that they hedge against them.

In fact, Orbis, a hedge fund included in the data, in its 1997 annual report writes "... [one of Orbis’ funds] was designed to, and did, offer members protection from losses stemming from a decline in overall world stockmarkets..."
Figure 3: Time series of equity and hedge fund indices. This figure plots the log-returns of the Latinamerica region (LA), the Europe region (EU), the Asian region (AS) and the CJUU region. It also plots the equally-weighted hedge fund index (HF). The figure indicates with gray bars recent large declines in overall world equity markets.

8. Empirical results

In this section, I first estimate the parameters \( \{K, j, r, \lambda\} \) using two models. Model I estimates these parameters from joint equally-weighted hedge fund index and international equity indices data. That is, I estimate the jump parameters using moment conditions (5.5) and (5.8). Model II estimates these parameters from international equity indices only. That is using moment conditions (5.8) alone. Then, I complete a series of robustness checks.
8.1. Parameter estimates

Table 7 reports the parameter estimates and their standard errors using the generalized method of moments. From Panel A of this table, Model I, we see that the estimated value of $\lambda = 0.74$, and this is significantly different from 0 at the 95% confidence level. That is, there is statistical evidence of correlated negative jumps.

The scaling jump parameters, $j$, are all positive and significantly different from 0. The average of the scaling jump parameter across regions is 0.21. The $\chi^2$ test for goodness-of-fit suggests that the model is not misspecified. The model has a value of $\chi^2 = 17.31$. Thus, the model cannot be rejected at the 95% confidence level. That is, the moment conditions (5.5) estimated from hedge fund data cannot be rejected.

We observe that the estimated value of $\lambda$ for Model I is larger than the one we obtain from asset returns alone, Model II. In addition, the average of the jump size scaling parameters from the assets alone, Model I, is smaller than the average size obtained from Model II.

We can also see that the optimal exposure to jumps, $y = 0.11$, from the joint time-series of assets and portfolio returns, Model I, is much smaller than the optimal exposure, $y = 0.27$, implied by the time-series of asset returns alone, Model II.

One possible interpretation of this last result is the peso problem: actual realizations of big jumps are too infrequent to be consistently observed and reflected in estimates drawn from the time series of asset returns alone but global equity-oriented hedge funds may account for the risk of simultaneous market crashes.

Moreover, the standard deviations of the estimates using both asset and portfolio
data, Model I, are much smaller than those of the estimates from asset returns alone, Model II. This result is in accordance to the Monte Carlo study in Section 6.

I am estimating an infinite activity Lévy process. The interpretation of the parameters of this process is not the same as the interpretation of the parameters of a jump-diffusion process. The main difference is that an infinite activity Lévy process exhibits an infinite number of small jumps in any finite time interval while for a compound Poisson process there is a finite number of jumps in any finite time interval.

Loosely speaking, a pure jump Lévy process is a Poisson jump process plus a process that exhibits an infinite number of tiny jumps. That is, a pure jump Lévy process, $Y_t$, can be decomposed as

$$Y_t = \sum_{s \leq t} \Delta Y_s I_{(\Delta Y_s \leq \varepsilon)} + \sum_{s \leq t} \Delta Y_s I_{(\Delta Y_s > \varepsilon)}$$  (8.1)

where $\Delta Y_s = Y_s - Y_{s-}$ with $Y_{s-} = \lim_{t \to s} Y_t$. The second term of equation (8.1) is a compound Poisson jump with intensity $\int_{z > \varepsilon} \nu (dz)$.

Using the decomposition for Lévy jumps (8.1), I plot in Figure 4 a mapping between jump sizes, $\varepsilon$, and the frequency of the jumps, $1/\int_{z > \varepsilon} \nu (dz)$. For example, the parameter estimates from joint international equity and hedge fund data, Model I, imply that on average a jump of 15%, or larger, on international equity indices is expected every 4 years. On the other hand, the parameter estimates from asset returns alone, Model II, imply that on average this jump is expected every 8 years.
Table 7: Parameter estimates.

Panel A: From Equity and Hedge Fund Returns (Model I)

<table>
<thead>
<tr>
<th></th>
<th>CJUU</th>
<th>LA</th>
<th>EU</th>
<th>AS</th>
<th>Avg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.7362</td>
<td></td>
<td></td>
<td></td>
<td>Avg.</td>
</tr>
<tr>
<td></td>
<td>(0.2455)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j$</td>
<td>0.1459</td>
<td>0.4272</td>
<td>0.1426</td>
<td>0.1310</td>
<td>0.2117</td>
</tr>
<tr>
<td></td>
<td>(0.0080)</td>
<td>(0.0243)</td>
<td>(0.0100)</td>
<td>(0.0109)</td>
<td>Riskless</td>
</tr>
<tr>
<td>$\bar{\omega}$</td>
<td>0.2916</td>
<td>0.0916</td>
<td>0.1254</td>
<td>0.0491</td>
<td>0.4423</td>
</tr>
<tr>
<td></td>
<td>(0.0589)</td>
<td>(0.0209)</td>
<td>(0.0346)</td>
<td>(0.0226)</td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td>0.1059</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0085)</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>17.31</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Panel B: From Equity Returns (Model II)

<table>
<thead>
<tr>
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<th>EU</th>
<th>AS</th>
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</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.3810</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.2244)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j$</td>
<td>0.1439</td>
<td>0.4194</td>
<td>0.1407</td>
<td>0.1313</td>
</tr>
<tr>
<td></td>
<td>(0.0154)</td>
<td>(0.0431)</td>
<td>(0.0201)</td>
<td>(0.0246)</td>
</tr>
<tr>
<td>$\bar{\omega}$</td>
<td>0.5773</td>
<td>0.1797</td>
<td>1.8302</td>
<td>-1.1335</td>
</tr>
<tr>
<td></td>
<td>(1.7925)</td>
<td>(0.6725)</td>
<td>(1.6307)</td>
<td>(0.8824)</td>
</tr>
<tr>
<td>$y$</td>
<td>0.2671</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.1889)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>11.37</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This table reports the estimates of the jump parameters and, in parenthesis, their standard errors. \{\lambda, j\} are obtained by GMM. Panel A gives the estimates from joint asset and portfolio returns, Model I, and Panel B gives the estimates from asset returns alone, Model II. In addition to the parameter estimates, the table reports the $\chi^2$ statistics to test the goodness-of-fit for the model. Also, I report the estimated values for the optimal exposure to jumps, $y$, the proportion of wealth invested in each region, $\bar{\omega}$, and in the riskless asset.
Figure 4: Mapping between magnitude and frequency of jumps from estimates. This figure plots the expected frequency of jumps implied by the estimates from Model I (solid line) and from Model II (solid line). A 95% confidence interval for Model I is provided. On average, a jump in equity indices larger than "Jump size" is expected every "Year".

8.2. Robustness checks

In this subsection, I examine whether the results reported in Table (7) are sensible to (1) the hedge fund returns data and (2) the utility function and Lévy measure.

8.2.1. To hedge fund data

Here I estimate the jump parameters but now using the value-weighted hedge fund index instead of the equally-weighted hedge fund index. The parameter estimates are presented in Table 8.

The implied exposure to jumps is almost the same as the one in Panel A of Table 7, in which I use the equally-weighted hedge fund index. The scaling jump factors, $j$, 

36
Table 8: Parameter estimates using value-weighted hedge fund index.

<table>
<thead>
<tr>
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<th>CJUU</th>
<th>LA</th>
<th>EU</th>
<th>AS</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ</td>
<td>0.6823</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.2434)</td>
<td></td>
<td></td>
<td>Avg.</td>
</tr>
<tr>
<td>j</td>
<td>0.1455</td>
<td>0.4281</td>
<td>0.1459</td>
<td>0.1143</td>
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<tr>
<td></td>
<td>(0.0084)</td>
<td>(0.0258)</td>
<td>(0.0113)</td>
<td>(0.0140)</td>
</tr>
<tr>
<td>ω</td>
<td>0.3432</td>
<td>0.0634</td>
<td>0.1182</td>
<td>-0.0163</td>
</tr>
<tr>
<td></td>
<td>(0.0855)</td>
<td>(0.0350)</td>
<td>(0.0602)</td>
<td>(0.0383)</td>
</tr>
<tr>
<td>y</td>
<td>0.0924</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0126)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>χ²</td>
<td>16.64</td>
<td></td>
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</tr>
</tbody>
</table>

This table reports the estimates of the jump parameters and, in parenthesis, their standard errors. \{λ, j\} are obtained by GMM. This table gives the estimates from joint asset and value-weighted hedge fund index log-returns. In addition to the parameter estimates, the table reports the $\chi^2$ statistics to test the goodness-of-fit for the model. Also, the estimated values for the exposure to jumps, y; and the proportion of wealth invested in each region, ω; and in the riskless asset are reported.

λ and y mainly remain unaffected. Moreover, the standard deviations of the estimates still is significantly smaller than those of model II.

8.2.2. To utility function and Lévy measure

Also, I estimate the model parameters but now using a different utility function and a different Lévy measure. The utility function that I use for this robustness analysis
is the log utility and for the Lévy measure I employ

\[ \nu(dz) = -\lambda \theta (z = -1) \, dz \]  \hspace{1cm} (8.2)

where \( \lambda \) is again a positive parameter.

Given this Lévy measure, the Lévy process becomes a jump-diffusion process. With the log utility function and this Lévy measure one can obtain the optimal weights, again, in closed-form. We should not expect the same parameter values as the ones in the previous subsections because here I am using a different model.

However, the implications of the parameter estimates using this log model are, indeed, the same as the ones implied by the power utility model. Namely, the estimated value of the exposure to jumps, \( y \), from joint assets and portfolio data is smaller than the estimated value of the exposure to jumps from assets alone. Also, the standard errors of the parameter estimates are much smaller when using both portfolio and asset returns. Table 9 reports the results.

9. Conclusions

Estimating the probability distribution of large and correlated jumps in asset returns is impractical because these jumps, such as observed during simultaneous market crashes, are too infrequent. This paper proposes to use portfolio returns as a means of generating additional information about the jumps’ characteristics.

Via an economic model of optimal demand, I propose a generalized method of moments procedure to estimate the probability distribution of jumps in asset returns
Table 9: Parameter estimates for the log model.

Panel A: From Equity and Hedge Fund Returns

<table>
<thead>
<tr>
<th></th>
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<th>LA</th>
<th>EU</th>
<th>AS</th>
<th>Avg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.6453</td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td>(0.2089)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j$</td>
<td>0.1335</td>
<td>0.3557</td>
<td>0.1227</td>
<td>0.1094</td>
<td>0.1803</td>
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<tr>
<td></td>
<td>(0.0142)</td>
<td>(0.0288)</td>
<td>(0.0107)</td>
<td>(0.0089)</td>
<td></td>
</tr>
<tr>
<td>$\bar{\omega}$</td>
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<tr>
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<td>(0.0208)</td>
<td>(0.0345)</td>
<td>(0.0228)</td>
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</tr>
<tr>
<td>$y$</td>
<td>0.0924</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0097)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>16.07</td>
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</tbody>
</table>

Panel B: From Equity Returns

<table>
<thead>
<tr>
<th></th>
<th>CJUU</th>
<th>LA</th>
<th>EU</th>
<th>AS</th>
<th>Avg.</th>
</tr>
</thead>
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<tr>
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<td></td>
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<td>0.3163</td>
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<td>0.1563</td>
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<td>(0.0306)</td>
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<td>-2.0862</td>
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<td>(3.2651)</td>
<td>(1.7176)</td>
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<tr>
<td>$y$</td>
<td>0.3910</td>
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<tr>
<td></td>
<td>(0.2425)</td>
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<tr>
<td>$\chi^2$</td>
<td>11.33</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This table reports the estimates of the jump parameters and, in parenthesis, their standard errors for the log utility model. \{\lambda, j\} are obtained by GMM. Panel A gives the estimates from joint asset and portfolio returns and Panel B gives the estimates from asset returns alone. In addition to the parameter estimates, the table reports the $\chi^2$ statistics to test the goodness-of-fit for the model. Also, I report the estimated values for the optimal exposure to jumps, $y$; and the proportion of wealth invested in each region, $\bar{\omega}$, and in the riskless asset.
from the joint observations of optimal portfolio and assets returns. I apply this method to estimate the probability distribution of simultaneous jumps in returns on international equities using international equity indices and hedge fund returns. Hedge fund returns serve as proxy for the optimal portfolio returns.

Finite sample evidence suggests that jump parameters estimated from both asset and optimal portfolio returns are unbiased and significantly more efficient than those estimated from asset returns alone. One average, the exposure of hedge funds to international jumps is smaller than the estimated value of the optimal exposure to jumps implied by international equities alone.

In future work, it would be interesting to incorporate correlated jumps not only in the form of simultaneous jumps, but also in a way which captures that a jump in one asset can cause an increase in the likelihood that a different jump will occur in that asset or in another asset. Self-exciting or mutual-exciting jump processes constitute a very promising approach to capture this effect. A method similar to the one proposed in this paper could be then used to estimate these processes.
References


