The Factor-Spline-GARCH Model for High and Low Frequency Correlations*

Robert F. Engle
Stern School of Business, New York University
rengle@stern.nyu.edu

Jose Gonzalo Rangel
Stern School of Business, New York University
jrange1@stern.nyu.edu

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Abstract

We propose a new approach to model and estimate the correlation structure in equity markets. We characterize high and low frequency components of the correlation matrix by combining a factor model with other specifications capturing dynamic properties of volatilities and covariances between a single common factor and idiosyncratic returns. Spline-GARCH and DCC models describe the dynamic behavior of these terms. We examine the relevance of the underlying assumptions in a simple one-factor CAPM model through an empirical evaluation of a number of correlation specifications from a range of factor models. Adding dynamic properties to the idiosyncratic terms suggests remarkable improvements in fitting the data. Our factor specification is consistent with the behavior of correlations in the US market. It also shows a strong performance in anticipating correlations, especially at long horizons.

Key Words: Factor models, Low frequency correlation, Dynamic conditional correlation, Spline, Generalized autoregressive conditional heteroskedasticity, Time varying betas.

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I. Introduction

Understanding the dynamics of correlations in financial markets is crucial to many important issues in finance. Optimal portfolio decisions, assessments of risks, hedging and pricing derivatives are examples of questions in financial decision making and financial regulation that require accurate measures and competent forecasts of comovements between asset returns. This paper introduces a new approach to characterize high and low frequency variation in equity correlations and to describe short and long term correlation behavior. By separating these term components, our method not only facilitates the economic interpretation of changes in the correlation structure but it also suggests measurement and forecast improvements.

A number of multivariate time series models have been proposed to capture the dynamic properties in the comovements of returns. As natural generalizations, multivariate versions of the well known univariate GARCH and Stochastic Volatility models guided the initial specifications (see, for example, Bollerslev, Engle and Wooldridge, 1988 and Harvey, Ruiz and Shephard, 1994). These initial generalizations proven to be limited because they are richly parameterized and/or difficult to estimate or because they are unattractive to describe empirical features of the data, for example constant correlation models (see Bollerslev 1990, Alexander, 1998, Harvey et al, 1994). Only recently, Engle (2002) introduced the Dynamic Conditional Correlation (DCC) model as an alternative approach to keep simplicity in the estimation process and parsimony in the dynamics of conditional correlations. However, none of the afore mentioned models provides a structural view of correlations that permits us to directly associate their variation through time with features of the underlying asset pricing models and/or of the fundamental economic variables.

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1 Recent surveys of multivariate GARCH and SV models are provided in Bowens, Laurent, and Rombouts (2003), Shephard (2004) and McAller (2005). Multivariate SV models are scant; recent developments include Assai and McAller (2005).
Financial structural models, on the other hand, have only recently introduced time variation in the correlation structure (e.g., Ang and Bekaert, 2002, Ang and Chen, 2002, Bekaert, Hodrick, and Zhang, 2005). Although these models are designed to link correlations with financial and economic variables, some assumptions are restrictive and a substantial part of the variation in the correlation structure remains unexplained.

This paper presents a model that incorporates both, the advantages of the time series approach to capture complex features of the data, and the attractiveness of financial structural models to identify different components of economic interest that drive the dynamics of variances and covariances. Based on a simple one-factor CAPM model for stock returns, we derive different specifications for asset correlations associated with different sets of assumptions in the underlying structural model. The effects of time varying betas and unobserved latent factors are addressed through specifications that incorporate dynamic patterns in the correlations between the market factor and each idiosyncratic component, as well as between each pair of idiosyncratic risks.

Furthermore, by disentangling dynamic features of market and idiosyncratic volatilities, we obtain a model that produces conditional correlations that mean revert toward a smooth time varying function, which proxies the low frequency component of correlations. Specifically, following the semiparametric approach of Engle and Rangel (2006), we identify the high and low frequency components of both systematic and idiosyncratic volatilities. Then we include these components in the specification of correlations, which also incorporates high frequency dynamics in the correlation among the factor and idiosyncratic innovations through a DCC process. The resulting "Factor-Spline-Garch" model generalizes the DCC model by allowing conditional correlations to mean revert toward a time varying function. Moreover, this approach allows us to set a link between economic determinants of volatilities and correlations.

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2 Latent factors have been used in multivariate settings. See for example Engle et al. (1990) in the GARCH context, and Diebold and Nerlove (1989) and Harvey et al. (1994) in the SV framework, and Andersen et al. (2001) in the realized variance context. Factor models with time varying betas have been studied in Bos and Newbold (1984), Ferson and Harvey (1991, 1993, 1999), and Ghysels (1998), among others.
From the empirical perspective, this study analyzes high and low frequency correlation patterns in the US market by considering the DJIA stocks. Our results support a negative relationship between correlations and idiosyncratic volatilities. Moreover, consistent with the findings of Campbell, Lettau, Malkiel and Xu (2001), our results suggest gradual increases on average idiosyncratic volatilities until the end of the nineties; however, we find a remarkable downward trend in these volatilities after 2000. These findings explain why in spite of dramatic declines in market volatility during the last years, correlations have declined only moderately. In addition, we find evidence that among the class of one-factor CAPM models, assumptions that allow dynamic behavior of idiosyncratic volatilities improve substantially the empirical performance of such models.

Overall, by adding dynamic features in a standard one-factor model without affecting its economic essence, the Factor-Spline-GARCH model provides a more general specification to describe the correlation structure of equity returns facilitating its economic interpretation. Although we take a time series approach here, our model introduces a framework that permit us to incorporate economic variables to explain and forecast equity correlations. This is encouraging given the strong forecasting performance shown by this model, especially at long horizons.

The paper is organized as follows: Section two provides a description of a number of correlation specifications associated with different assumptions in the factor setup. Section three introduces the Factor-Spline-GARCH model and discusses estimation issues. Section four presents an empirical analysis of the US market. It includes an empirical evaluation of correlation specifications derived from different factor models. Section five examines the forecast performance of the Factor-Spline-GARCH model, and Section six concludes.

II. A Single Factor Model and Return Correlations

In this section, we present a simple version of the APT asset pricing model of Ross (1976) and we describe how modifying its underlying assumptions changes the implied
correlation structure. Suppose that there is a single market factor that enters linearly in the pricing equation such as in the CAPM model. Under this model, the specification for the excess return of asset $i$ is:

$$r_i - r^f = \beta_i (\tilde{r}_m - r^f) + u_i,$$

where $r_m = (\tilde{r}_m - r^f)$ denotes the common factor market return and $r^f$ represents the risk free rate. The first term characterizes asset $i$'s systematic risk and the second describes its idiosyncratic component. The standard APT structure assumes constant betas, idiosyncrasies uncorrelated with the factor(s) and idiosyncrasies uncorrelated with each other:

$$E(r_m u_i) = 0, \quad \forall i,$$  \hspace{1cm} (2)

$$E(u_i u_j) = 0, \quad \forall i \neq j$$ \hspace{1cm} (3)

Thus, the assumptions in the factor structure impose a restriction in the covariance matrix of returns. Under these standard assumptions a typical element of the unconditional covariance and correlation matrices can be respectively characterized as:

$$\text{cov}(r_i, r_j) = \beta_i \beta_j \sigma_m^2 + \begin{cases} \sigma_i^2 & i = j \\ 0 & i \neq j \end{cases}$$  \hspace{1cm} (4)

$$\text{corr}(r_i, r_j) = \frac{\text{cov}(r_i, r_j)}{\sqrt{\text{var}(r_i)} \sqrt{\text{var}(r_j)}} = \frac{\beta_i \beta_j \sigma_m^2}{\sqrt{\beta_i^2 \sigma_m^2 + \sigma_i^2} \sqrt{\beta_j^2 \sigma_m^2 + \sigma_j^2}}.$$  \hspace{1cm} (5)

where $\sigma_m^2$ and the $\sigma_i^2$'s are the variances of the factor and the idiosyncratic terms, respectively. Now, from the definition of conditional correlation,

$$\rho_{i,j,t} = \text{corr}(r_i, r_j) = \frac{E_{t-1}((\beta_i r_{mt} + u_{it})(\beta_j r_{mt} + u_{jt}))}{\sqrt{E_{t-1}((\beta_i r_{mt} + u_{it})^2) E_{t-1}((\beta_j r_{mt} + u_{jt})^2)}},$$  \hspace{1cm} (6)

and assuming that the moment restrictions in (2) and (3) hold conditionally, we obtain:

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3 The risk free rate term has been dropped to simplify notation.
This expression suggests that the dynamic behavior of conditional correlations is determined exclusively by dynamic patterns in the conditional variances of market and idiosyncratic risks. The betas determine sign and location.

In this single factor model, if the restriction in (2) holds conditionally, then the betas are constant and correctly estimated from simple time series regressions of asset returns on the market portfolio. Restriction (3) rules out correlation between idiosyncratic innovations, which precludes the possibility of missing pricing factors in the model. These restrictions are empirically unappealing and limit importantly the dynamic structure of correlations. Allowing for temporal deviations from such conditions increases substantially the flexibility of the resulting correlation models to capture empirical features of the data without affecting the economic essence of the factor model. The following proposition characterizes changes in the specification of correlations when such restrictions are relaxed.

**Proposition 1**: Consider the model specification in Equation (1) and let \( \mathcal{Z}_t \) denote the set of current and past information available in the market. a) Suppose that \( E(r_{mt}u_t) = 0 \), but \( E_{t-1}(r_{mt}u_t) \neq 0 \), then the correlation structure corresponds to that of a single factor model with time varying betas satisfying \( \beta_t = \beta_i + w_t \), and the following assumptions:

i) \( w_t = \{w_{1t}, w_{2t}, \ldots, w_{N_t}\} \) is a zero mean covariance-stationary process.

ii) \( \{w_{it}, w_{jt}\} \) is a martingale difference sequence with respect to \( \mathcal{Z}_t \), \( \forall i \neq j \).

iii) Letting \( \tilde{u}_{it} \) denote the idiosyncratic terms, \( \{\tilde{u}_{it}, \tilde{u}_{jt}\} \) and \( \{r_{mt}w_{jt}, \tilde{u}_{jt}\} \) are martingale difference sequences with respect to \( \mathcal{Z}_t \), \( \forall i \neq j \).

iv) \( E_{t-1}(r_{mt} \mid w_t) = E_{t-1}(r_{mt}) \).
v) \( E(w_{it} r_{mt}^2) = 0, \ \forall i = 1, \ldots, N. \)

Such correlation structure can be described in terms of the conditional correlation as follows:

\[
\rho_{i,j,t} = \frac{\beta_j E_{t-1}(r_{mt}^2) + \beta_j E_{t-1}(r_{mt} u_{it}) + \beta_i E_{t-1}(r_{mt} u_{jt})}{\sqrt{\beta_i^2 E_{t-1}(r_{mt}^2) + E_{t-1}(u_{it})^2} \sqrt{\beta_j^2 E_{t-1}(r_{mt}^2) + E_{t-1}(u_{jt})^2} + 2 \beta_j E_{t-1}(r_{mt} u_{jt})} \tag{8}
\]

Moreover, b) suppose that we allow \( E_{t-1}(u_{it} u_{jt}) \neq 0, \ \forall i, j, \) by relaxing ii) and/or iii), then the correlation structure corresponds to that in a model with more than one factor. Such specification takes the following form:

\[
\rho_{i,j,t} = \frac{\beta_j E_{t-1}(r_{mt}^2) + \beta_j E_{t-1}(r_{mt} u_{it}) + \beta_i E_{t-1}(r_{mt} u_{jt}) + E_{t-1}(u_{it} u_{jt})}{\sqrt{\beta_i^2 E_{t-1}(r_{mt}^2) + E_{t-1}(u_{it})^2} \sqrt{\beta_j^2 E_{t-1}(r_{mt}^2) + E_{t-1}(u_{jt})^2} + 2 \beta_j E_{t-1}(r_{mt} u_{jt})} \tag{9}
\]

The proof is given in Appendix A1. Proposition 1 motivates our econometric approach, which will be introduced in the following section. It provides not only an econometric estimation strategy for conditional and unconditional correlations (which guarantees positive semi-definiteness of the correlation matrix), but also a framework to test the empirical importance of each assumption in our simple factor model.

### III. An Econometric Model for Structural Correlations

Different parametrizations of each component in the correlation structure implied by a factor model and its underlying assumptions lead to different models for correlations. This section begins by motivating our econometric specification. We then present a new structural model for high and low frequency correlations and discuss the estimation strategy.

\[\text{Assumptions } i) \text{ and } v) \text{ guarantee that } E(\beta_{it}) = \beta_i = \frac{\text{cov}(r_{it}, r_{jt})}{\text{var}(r_{it})}.\]

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4 Assumptions i) and v) guarantee that \( E(\beta_{it}) = \beta_i = \frac{\text{cov}(r_{it}, r_{jt})}{\text{var}(r_{it})}. \)
The Factor-Spline-GARCH Model

The evolution of equity volatilities over time shows different patterns at different frequencies. Short term volatilities are mainly determined by news arrivals, which induce price changes at very high frequencies. Longer term volatilities show patterns governed by slow moving structural economic and financial variables. Engle and Rangel (2006) analyze such determinants and characterize the dynamic behavior of equity volatilities at low frequencies. They find economically and statistically significant variation in the US low frequency market volatility as well as in most cases in developed and emerging countries. We introduce this effect in our model for structural correlations by including an equation that describes the dynamic behavior of this low frequency market factor volatility.

In regard to idiosyncratic volatilities, incorporating their low frequency variation in the correlation structure is also appealing from the empirical and theoretical perspective since these low frequency components describe long term trends in volatilities. For instance, in an influential study, Campbell, Lettau, Malkiel and Xu (2001) find evidence of a positive deterministic trend in idiosyncratic firm-level volatility. Moreover, they find that the market volatility does not observe such increasing deterministic trend, which suggests a declining long term effect in the correlations among individual stocks. Theoretical explanations for the upward trend in idiosyncratic volatilities have been associated with different firm features such as the level and variance of total firm profitability, age, institutional ownership, and the level and variance of growth options available to managers (see Pastor and Veronesi (2003), Wei and Zhang (2004), and Cao, Simin and Zhao (2006)). In a global context, Engle and Rangel (2006) find that country specific macroeconomic variables and the financial development of individual exchanges explain the time and cross sectional variation in the long term country-specific volatilities. These empirical and theoretical results show the importance of considering long term patterns of systematic and idiosyncratic volatilities in the correlation structure.

5 For different approaches on low frequency economic determinants of stock market volatility see, for example, Officer(1973), Schwert(1989), and Engle, Ghysels and Sohn (2006).
From the econometric point of view, the Spline-GARCH model of Engle and Rangel (2006) provides a semi-parametric framework to separate high and low frequency components of volatilities. Following this approach, we model the market factor in Equation (1) as:

\[ r_{mt} = \sqrt{\tau_{mt} g_{mt}} e_{tm}, \text{ where } e_{tm} | \Phi_{t-1} \sim (0,1) \]

\[ g_{mt} = (1-a_m-b_m) + a_m \left( \frac{r_{mt-1}}{\tau_{mt-1}} \right)^2 + b_m g_{mt-1} \]

\[ \tau_{mt} = c_m \exp \left( w_m t + \sum_{j=1}^{k_m} w_{mj} \left( (t-t_{j-1})_+ \right)^2 \right), \quad (10) \]

and the idiosyncratic returns as:

\[ \mu_{it} = \sqrt{\tau_{it} g_{it}} e_{it}, \text{ where } e_{it} | \Phi_{t-1} \sim (0,1) \]

\[ g_{it} = (1-a_i-b_i) + a_i \left( \frac{r_{it-1} - \beta_i r_{mt-1}}{\tau_{it-1}} \right)^2 + b_i g_{it-1} \]

\[ \tau_{it} = c_i \exp \left( w_{it} t + \sum_{r=1}^{k_i} w_{ir} \left( (t-t_{r-1})_+ \right)^2 \right), \quad \forall i \quad (11) \]

where \( g_{mt} \) and the \( g_{it} \)'s characterize the high frequency market and idiosyncratic volatilities, whereas \( \tau_{mt} \) and the \( \tau_{it} \)'s describe the low frequency market and idiosyncratic volatility components, respectively. The number of knots, \( k_i \)'s, can be selected optimally using an information criterion. As in Engle and Rangel (2006), we will use the Schwarz Criterion and Gaussian innovations.\(^6\)

Following the discussion in the previous section and Proposition 1, we incorporate time varying correlations between the market factor and idiosyncratic returns, as well as among the idiosyncratic terms themselves. Specifically, we assume that the vector of innovations in Equations (10) and (11), \( (e_{tm}, e_{1t}, e_{2t}, ..., e_{Nt})' \), follows the DCC model of Engle (2002). Note that all the elements in this vector have unit conditional variance.

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\(^6\) We follow the same notation as in Engle and Rangel (2006), where \( (t - x)_+ = (t - x) \text{ if } t > x \), and it is zero otherwise. We refer to the original paper for further details on the spline specification.
Thus from the second stage in the standard DCC model, these correlations can be written as:

\[
\rho_{i,j,t}^e = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}} \sqrt{q_{j,j,t}}},
\]

\[
q_{i,j,t} = \bar{\rho}_{i,j,t} + a_{DCC}(e_{i,j,t-1} - \bar{\rho}_{i,j,t}) + b_{DCC}(q_{i,j,t-1} - \bar{\rho}_{i,j,t}), \quad \forall i, j \in \{1, \ldots, N\},
\]

\[
q_{m,i,t} = \bar{\rho}_{m,i,t} + a_{DCC}(e_{m,i,t-1} - \bar{\rho}_{m,i,t}) + b_{DCC}(q_{m,i,t-1} - \bar{\rho}_{m,i,t}), \quad \forall i \in \{m, 1, \ldots, N\},
\]

where \( \bar{\rho}_{i,j,t} = E(e_{i,t}, e_{j,t}) \) and \( \bar{\rho}_{m,i,t} = 1 \), for all \( i=m,1,2,\ldots,N \). Moreover, given the time variation in betas described in Proposition 1, we assume that \( \bar{\rho}_{m,j,t} = 0 \), for all \( i=1,2,\ldots,N \).

The specifications above along with the factor structure presented in Section II constitute the full Factor-Spline-GARCH (FSG-DCC) model and its correlation structure is described in the following proposition.

**Proposition 2:** Given a vector of returns \( (r_{i,t}, r_{2,t}, \ldots, r_{N,t})' \) satisfying the factor structure in Equation (1), suppose that the common market factor \( r_{m,t} \) is described by (10), the idiosyncratic term \( u_{i,t} \) follows the process in (11), for all \( i=1,2,\ldots,N \), and the vector of innovations \( (e_{1,t}, e_{2,t}, \ldots, e_{N,t})' \) follows the DCC process in (12) and its assumptions, then the high frequency (conditional) correlation between \( r_{i,t} \) and \( r_{jt} \) is given by:

\[
\rho_{i,j,t} = \frac{\beta_x \beta_j \tau_{m,t} g_{m,t} + \beta_i \sqrt{\tau_{m,t}} g_{m,t} \sqrt{\tau_{j,t}} g_{j,t} \rho_{m,j,t}^e + \beta_j \sqrt{\tau_{m,t}} g_{m,t} \sqrt{\tau_{j,t}} g_{j,t} \rho_{m,j,t}^e + \beta_i \sqrt{\tau_{j,t}} g_{j,t} \sqrt{\tau_{j,t}} g_{j,t} \rho_{m,j,t}^e + \beta_j \sqrt{\tau_{m,t}} g_{m,t} \sqrt{\tau_{j,t}} g_{j,t} \rho_{m,j,t}^e}{\sqrt{\beta_i^2 \tau_{m,t} g_{m,t} + \tau_{j,t} g_{j,t} + 2 \beta_i (\tau_{m,t} g_{m,t} \tau_{j,t} g_{j,t}) \rho_{m,j,t}^e} \sqrt{\beta_j^2 \tau_{m,t} g_{m,t} + \tau_{j,t} g_{j,t} + 2 \beta_j (\tau_{m,t} g_{m,t} \tau_{j,t} g_{j,t}) \rho_{m,j,t}^e}},
\]

and, the low frequency component of this correlation is time varying and takes the following form:

\[
\rho_{i,j,t} = \frac{\beta_x \beta_j \tau_{m,t} g_{m,t} + \beta_i \sqrt{\tau_{m,t}} g_{m,t} \sqrt{\tau_{j,t}} g_{j,t} \rho_{m,j,t}^e + \beta_j \sqrt{\tau_{m,t}} g_{m,t} \sqrt{\tau_{j,t}} g_{j,t} \rho_{m,j,t}^e + \beta_i \sqrt{\tau_{j,t}} g_{j,t} \sqrt{\tau_{j,t}} g_{j,t} \rho_{m,j,t}^e + \beta_j \sqrt{\tau_{m,t}} g_{m,t} \sqrt{\tau_{j,t}} g_{j,t} \rho_{m,j,t}^e}{\sqrt{\beta_i^2 \tau_{m,t} g_{m,t} + \tau_{j,t} g_{j,t} + 2 \beta_i (\tau_{m,t} g_{m,t} \tau_{j,t} g_{j,t}) \rho_{m,j,t}^e} \sqrt{\beta_j^2 \tau_{m,t} g_{m,t} + \tau_{j,t} g_{j,t} + 2 \beta_j (\tau_{m,t} g_{m,t} \tau_{j,t} g_{j,t}) \rho_{m,j,t}^e}},
\]

\[\text{We consider a mean reverting DCC model. The parameters are positive and their sum is less than one. See Engle and Sheppard (2002) for details.}\]

\[\text{Idiosyncratic innovations can be seen as residuals from regressions of returns on the market factor. Although we account for time varying betas allowing for temporal variation in the conditional covariance between the factor and the idiosyncrasies, these residuals are still unconditionally uncorrelated with the factor.}\]
Moreover, assuming that $E_t(\tau_{k,t+h}) = \tau_{k,t}$, $\forall h > 0, k = 1,2,\ldots,N$, then Equation (14) is the long horizon forecast of $\rho_{i,j,t}$:

$$\lim_{h \to \infty} \rho_{i,j,t+h} = \overline{\rho}_{i,j,t}$$

The proof is given in Appendix A2. Note that Equation (13) is a parametrized version of Equation (9) in Proposition 1. Equation (14) approximates the slow moving component of correlation, which can be associated with long term correlation dynamics. Indeed, the high frequency correlation parsimoniously mean reverts toward this time varying low frequency term. This approximation can be improved by allowing time variation in the unconditional (or low frequency) correlation between the innovation terms in the DCC model. We leave this extension for future research.

Equation (14) provides a simple approach to anticipate long run correlations using economic variables. Specifically, we only need forecasts of low frequency market and idiosyncratic volatilities, and such forecasts can be easily obtained from univariate models that incorporate economic variables. For instance, the approach of Engle and Rangel (2006) permits us to construct forecasts of the market volatility using macroeconomic and market information; indeed, maintaining for simplicity idiosyncratic low frequency volatilities constant at their last value during the forecast period, we can use such economic market volatility forecasts to obtain long run correlation forecasts based on macroeconomic information.

Moreover, from a time series perspective, Equation (15) presents a useful forecasting relation in which, due to the mean reversion properties of the model, the time varying low frequency correlation can be interpreted as the long run correlation forecast under the assumption that the low frequency market and idiosyncratic volatilities stay constant during the forecast period. This provides a time series approach where long term forecasts are constructed from information in the betas and past returns.
Another interesting case imbedded in Proposition 2 occurs when the low frequency components of volatility are constant over both the estimation and forecast periods. This restricted version can be defined as the Factor-GARCH-DCC (FG-DCC) model and it is derived by assuming $\tau_{m,j} = \sigma_m^2$ and $\tau_{i,j} = \sigma_i^2, \forall i$, in (10) and (11). The corresponding variance specifications for the factor and the idiosyncrasies become standard mean reverting GARCH(1,1) processes, which can be respectively written as:

$$h_{mt} \equiv \sigma_m^2 g_{mt} = \sigma_m^2 (1 - a_m - b_m) + a_m (r_{mt-1})^2 + b_m h_{m,t-1}, \quad (16)$$

and

$$h_{it} \equiv \sigma_i^2 g_{it} = \sigma_i^2 (1 - a_i - b_i) + a_i (r_{it-1} - \beta_i r_{mt-1})^2 + b_i h_{i,t-1}, \forall i = 1, 2, \ldots, N. \quad (17)$$

Hence, the conditional correlation in Equation (13) becomes:

$$\rho_{i,j,t} = \frac{\beta_i \beta_j h_{mt} + \beta_i \sqrt{h_{mt}} \sqrt{h_{it}} \rho_{m,i,t} + \beta_j \sqrt{h_{mt}} \sqrt{h_{jt}} \rho_{m,j,t} + \sqrt{h_{mt}} \sqrt{h_{jt}} \rho_{i,j,t}}{\sqrt{\beta_i^2 h_{mt} + h_{it} + 2 \beta_i (h_{mt} h_{it})^{1/2} \rho_{m,i,t} \sqrt{\beta_j^2 h_{mt} + h_{jt} + 2 \beta_j (h_{mt} h_{jt})^{1/2} \rho_{m,j,t}}}, \quad (18)$$

and the low frequency correlation is the following constant:

$$\overline{\rho}_{i,j} = \frac{\beta_i \beta_j \sigma_m^2 + \sigma_i \sigma_j \overline{\rho}_{i,j}}{\sqrt{\beta_i^2 \sigma_m^2 + \sigma_i^2 \sqrt{\beta_j^2 \sigma_m^2 + \sigma_j^2}}}. \quad (19)$$

This equation also represents the long run correlation forecast associated with the FG-DCC model. In section V, we will evaluate the forecast performance at long horizons of the two factor specifications presented above and a standard DCC model.

**Estimation**

The FSG-DCC model can be specified using matrix notation. Suppose we have a vector of returns in excess of the risk free rate, including the market factor, at time $t$:

$$r_t = (r_m, r_{1t}, \ldots, r_{nt})'$$

The system of equations in the factor setup can be written as:

$$r_t = Bu_t, \quad (20)$$
where $u_t$ contains the market factor and the idiosyncrasies, $u_t = (r_m, u_{1t}, u_{2t}, ..., u_{Nt})'$, $B = \begin{pmatrix} 1 & 0_{1 \times N} \\ \beta & I_{N \times N} \end{pmatrix}$, $\beta = (\beta_1, \beta_2, ..., \beta_N)'$, $0_{1 \times N}$ is an N-dimensional row vector of zeros, and $I_{N \times N}$ is the N-dimensional identity matrix.

Now, assuming

$$u_t \mid \mathcal{F}_{t-1} \sim N(0, H_t^u),$$

we can decompose the covariance matrix of $u_t$ following the approach in the standard DCC model. Hence,

$$H_t^u = D_t R_t D_t,$$

where $D_t = diag \left\{ \sqrt{\tau_{kt}}, h_{kt} \right\}$, $k = m, 1, 2, ..., N$. The standardized innovations in Proposition 2 are the elements of $D_t^{-1} u_t = (\varepsilon_t^m, \varepsilon_t^1, \varepsilon_t^2, ..., \varepsilon_t^N)'$. Furthermore, Equation (12) describes the elements of $R_t$.

Going back to the original vector of returns, we have that $r_t \mid \mathcal{F}_{t-1} \sim N(0, H_t)$, where the variance takes the following form:

$$Var(r_t) = H_t = BD_t R_t D_t B'^{-1}$$

Given that $B$ is a non-singular square matrix, we can decompose the likelihood as the sum of a “betas and volatility” part and a “correlation” part, just as in the DCC model. Therefore, the estimation strategy is the familiar two-stage approach, where the betas along with the Spline-Garch volatility parameters are estimated in a first stage, and the correlation parameters in Equation (12) are estimated in the second stage.

Regarding the second stage, the estimation can be performed in the usual way, which jointly estimates the whole correlation matrix. However, although DCC is very

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9 The correlation matrix of $r_t$ is $R_t = diag \left\{ BD_t R_t D_t B'^{-1} \right\}^{-1} BD_t R_t D_t B'^{-1} diag \left\{ BD_t R_t D_t B'^{-1} \right\}^{-1}$, and its typical element, ignoring the first row and first column that contains correlations between the factor and the idiosyncrasies, is the expression in Equation (13).

10 Consistency and asymptotic normality are satisfied under standard regularity conditions. See Engle (2002) and Newey and McFadden (1994).
parsimonious in its parameterization, it is biased and slow for large covariance matrices. Alpha is biased downward and may approach zero. Thus, estimated correlations are less variable in big systems than when estimated for subsets even for simulated data (see Engle and Sheppard (2005)). Since our empirical analysis includes a relatively large number of assets, we are interested in a simpler approach to estimating large covariance matrices. A suggested method is the MacGyver method introduced by Engle (2007). This method is designed to solve the mentioned problems and simplifies the estimation process for large systems. It can be easily implemented by following the next simple steps:

1. Estimate all bivariate models. Each should be consistent for the same parameters.
2. Select a single parameter from these estimates, preferably the median.
3. Calculate all the correlations.

**IV High and Low Frequency Correlations in the US Market.**

**Data**

We use daily returns on the DJIA stocks from December 1988 to December 2006. The data is obtained from CRSP. During this period, there have been a number of changes in the index, including additions, deletions, and mergers. We include all the stocks in the 2006 index and those in the 1989 index that could be followed over the sample period. As a result, we obtain a sample of 33 stocks. Regarding the market factor, we use daily returns on the S&P500.

Individual stocks are described in Table 1, which include company names, market tickers, average returns and average annualized volatility over the whole sample period. The most volatile stocks in the sample are INTC and HPQ, whereas the least volatile are

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11 Those include Chevron (CVX), Goodyear (GT), and International Paper (IP).
XOM, CVX, and 3M.\textsuperscript{12} The stocks with largest average daily return are MSFT and INTC and those with the smallest values are GM and IP.

**Estimation Results**

We estimate the FSG-DCC model following the two stage approach with bivariate DCC systems as described in Section III. Table 2 presents the results associated with each estimation stage. The parameters estimated in the first stage are those in Equations (10) and (11). For reasons of space, we present results only for the betas, $\{a_i, b_i\}, i = m, 1, 2, ..., N$, and the optimal number of knots associated with each spline, $\{k_i\}, i = m, 1, ..., N$.

The first column presents the market betas associated with each stock and Figure 1 illustrates their distribution. All of them are highly significant and their values go from 0.66 to 1.54. CVX and XOM show the lowest factor loadings and the lowest volatilities over the sample period (according to Table 1), whereas INTC and HPQ show the largest betas as well as the largest sample volatilities. The second column presents the estimates for the ARCH volatility coefficients. They take values between 0.01 and 0.19. The median is 0.07. Column 3 presents the estimates of the GARCH volatility effects. They are between 0.12 and 0.99. The median is 0.83. Stocks with low GARCH effects along with large values of the ARCH coefficient are associated with noisy patterns in its idiosyncratic component. For example, the first panel of Figure 2 illustrates the familiar highly persistent case associated with a small ARCH coefficient (0.02) and GARCH effects close to 1, whereas the last panel in this figure shows a noisier case where the ARCH affect is about 0.10 and the GARCH effect is only 0.5. The last column of Table 2 shows the optimal number of knots in the spline function. This number can be seen as an indicator of the variability in the long run trend of idiosyncratic volatility.

\textsuperscript{12} We use the ticker name to identify individual stocks. The full company names are presented in the first column of Table 1.
The bottom part of Table 2 presents the estimation results of the second stage. The first column reports the standard DCC estimates from the multivariate traditional approach along with their standard errors. The second column presents the MacGyver DCC coefficients, which are the median values of all DCC estimates from bivariate systems, as referred in step 1 of the MacGyver method. The estimate of $a_{DCC}$ goes from 0.0027 in the standard method to 0.005 in the bivariate approach. This suggests a correction in the downward bias of the traditional DCC estimator.

Typical patterns in the idiosyncratic volatilities are illustrated in Figure 2. A common feature is an increasing trend during the period 1989-1999 that is consistent with the results of Campbell et al (2001); however, this effect disappears after 2000 when the idiosyncratic volatility starts declining. This effect is reversed in some cases at the end of the sample (see graphs in the middle left and in the bottom left of Figure 2). The occurrence of nonlinear cyclical patterns in the low frequency volatilities is described by the number of knots in the spline function. For example, among the graphs in Figure 2, the first graph has the most parsimonious low frequency component. It shows a monotonic increasing trend before 1999 and a monotonic decline afterwards. As we move to the bottom of the figure, the number of knots increases and the cyclical effects are more frequent. Figure 3 shows the dynamic behavior of the high and low frequency components of market volatility. The downward trend observed at the end of the sample raises the question of whether correlations have followed the same pattern.

According to our factor structure, two increasing low frequency idiosyncratic volatilities would suggest a declining low frequency correlation (and vice versa), other things equal. When these idiosyncratic volatilities move in opposite directions the effect is not clear. For example, if we focus on the cases in Figure 2 and on the last years of the sample, where market volatility is decreasing, we can observe that the stocks showing increasing idiosyncratic volatilities are associated with declining low frequency correlations, as illustrated in Figure 4. In these examples market and idiosyncratic volatilities affect factor correlations in the same direction. In contrast, Figure 5 presents examples where market and idiosyncratic volatilities have opposite effects on low frequency correlations,
which show increasing and decreasing patterns during the last year of the sample. The effects become more complicated when we allow for time variation in the factor loadings.

While the previous examples illustrate particular cases of correlation patterns, the dimensionality of our results make difficult to analyze them at a disaggregated level. Instead, we present an aggregated analysis of idiosyncratic volatilities and correlations. Table 3 shows averages over the whole sample period of annualized low frequency volatilities along with low frequency and rolling correlations of each stock with respect to all the rest. These results are illustrated in Figure 6. The first panel compares average idiosyncratic volatilities with average low frequency correlations for each stock. Consistent with our previous examples, it is clear that assets with larger idiosyncratic volatilities tend on average to show lower levels of low frequency correlations. This negative relation is supported by the results in the second panel of Figure 6, where model free average rolling correlations also show a negative relation with the average low frequency idiosyncratic volatilities. In addition, Figure 7 illustrates how the average of low frequency idiosyncratic volatilities has decreased during the last years along with that of market volatility. This suggests that although market volatility has reached historical low values by the end of 2006, correlations have decreased only moderately. Indeed, the two year average rolling and FSG-DCC correlations presented in Figure 8 indicate that correlations over the period 2005-2006 were approximately 28% larger than those over the period 1993-1994, which is characterized by similar low levels of market volatilities. It is important to note that the previous results provide empirical support for a factor structure in the US market.

\[ \text{Avg. low freq. correlation of asset } i = \frac{1}{T} \left( \frac{1}{N-1} \sum_{j \neq i}^{T} \rho_{i,j,t} \right) \]

For average rolling correlations, we replace \( \rho_{i,j,t} \) by the corresponding rolling correlation \( \rho_{i,j,t}^{\text{Rolling}} \).

\[ \bar{\rho}_{p}^{\text{Rolling}} = \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{1}{T_{p}} \left( \frac{1}{N-1} \sum_{j \neq i}^{T_{p}} \rho_{i,j,t}^{\text{Rolling}} \right) \right\} \]

where \( T_{p} \) denotes the number of daily observations over the specific period, and \( \rho_{i,j,t}^{\text{Rolling}} \) is the rolling correlation between assets \( i \) and \( j \) at time \( t \).
Overall, the low frequency component of correlation illustrates the long term patterns observed in the correlation structure of US stocks. It proxies the process to which the high frequency component reverts and, as mentioned earlier, it can be seen as the long term forecast of conditional correlations. Moreover, comparing these two correlation components with the rolling correlations, we observe that they both provide a good description of the empirical patterns observed in the data. Although this is remarkable given the single factor specification, we have relaxed many of the assumptions in the simple model described in Equations (1)-(5). Now, which of these assumptions were more restrictive? We answer this question empirically by evaluating restricted correlation specifications in terms of different underlying assumptions in the simple one-factor framework.

**Evaluating the Assumptions**

In this section, we evaluate the range of factor models with varying dynamic components in terms of their empirical fit. For example, we consider the simplest case, labeled FC-C, where factor and idiosyncratic volatilities are constant over the sample period (at high and low frequencies), and restrictions in (2) and (3) apply. We estimate the correlations from this model and, in the spirit of the quasi-maximum likelihood approach, we compute the corresponding log likelihood assuming a Gaussian distribution. We then evaluate the log likelihood for models that relax one or more assumptions of the initial model. We obtain a set of models that can be compared in terms of their empirical fit and allow us to determine which restrictions are the most important.

The assumptions to be weakened are the following:

1. Constant low frequency volatilities.
2. Constant high frequency volatility of the market factor.
3. Constant high frequency volatility of the idiosyncratic component of returns.
4. Constant betas.
5. One single common factor.

For example, including low frequency variation in the volatilities of the factor and the idiosyncratic terms, using the spline functions defined in Section III, we obtain a specification called FS-S. Similarly, when high frequency GARCH dynamics is added to the factor volatility maintaining the idiosyncratic volatilities constant, we obtain the FG-C model. These models and their correlations, along with a range of specifications derived from adding dynamics to the previous assumptions, are described in Table 4. Their likelihoods are constructed from the general factor structure in Equation (20) assuming that \( r_\tau | \mathcal{F}_{\tau-1} \sim N(0, H_{\tau}) \). A mapping of each correlation specification in Table 4 with a specific covariance matrix provides the inputs to compute the log likelihood of each model. We estimate the models in the first panel of Table 4 using MLE. For the other models, we follow the two-stage DCC estimation approach described in Engle (2002).

Table 5 presents the log likelihoods of the factor models in Table 4, along with likelihood ratio tests that compare each model with the biggest FSG-DCC model. The results indicate that the FSG-DCC model dominates the other specifications. Close to this model is the FSG-IDCC model, which accounts for the effect of latent (missing) factors and both, high and low frequency dynamics in market and idiosyncratic volatilities. The FG-DCC model follows in the list. Therefore, although adding time variation in the betas improves the empirical fit, particularly when low frequency dynamics is incorporated, our results favor specification (9) over specification (8) in Proposition 1. The model with poorest empirical fit is the constant correlation model (FC-C). Overall, we find that specifications with low frequency dynamics dominate those with only high frequency dynamics. These results are maintained even if we penalize bigger models. Indeed, the last column of Table 5 presents the corresponding BIC statistics, which lead to the same conclusions.

In addition, the results indicate that the largest improvements in the likelihoods are achieved when we relax the assumption of constant low frequency market and
Idiosyncratic volatilities, the assumption of constant high frequency idiosyncratic volatilities, and when we incorporate the effect of latent factors. This suggests that, besides the importance of modeling market behavior, adding dynamic features to the idiosyncratic components in terms of their volatilities, or in terms of systematic patterns associated with the existence of unobserved latent factors, improves substantially the empirical performance of this class of one-factor CAPM models.

V Forecast Performance of the Factor-Spline-GARCH Model

We have introduced a model that describes different features of correlations associated with their short and long term dynamic behavior. It is natural to ask whether this type of model can improve forecasts of correlations. To address this question, we first illustrate the forecasting properties of the FSG-DCC model and then we evaluate its performance in terms of the performance of other competitive specifications using an economic loss function.

In terms of forecasting properties, we mentioned earlier that the conditional correlation in the FSG-DCC model mean reverts toward the smooth low frequency correlation component rather than toward a constant level (which is determined by the sample correlation in the DCC model or by Equation (19) in the FG-DCC model). Hence, given the empirical patterns in low frequency correlations discussed in Section III, long horizon forecasts from the FSG-DCC model might differ considerably from those based on models that mean revert to constant levels. Moreover, the FSG-DCC model tends to show faster mean reversion than its GARCH competitors since its volatility components have smaller persistence than volatilities estimated from standard GARCH models (see, Engle and Rangel, 2006). These two characteristics are illustrated in the examples of Figure 9, which show multiple-step ahead forecasts associated with different correlation specifications. Here, we compare forecasting features of the FSG-DCC model described in Equations (13) and (14) with those of two competitors: 1) the standard DCC model and 2) the restricted FG-DCC model in Equation (18). The first case is interesting because it provides a non-factor reference to evaluate the importance of imposing a factor structure.
The second case is relevant to evaluate the implications of allowing time varying low frequency volatilities. The forecasts are constructed out-of-sample at horizons that go from 1 to 130 steps ahead. As mentioned before, the low frequency volatility forecasts in the FSG-DCC model are constructed under the assumption that $\tau_{i,j+k|t} = \tau_{i,j|t}$, for all $i=m,1,2,\ldots,N$, and $k=1,2,\ldots,120$.

The examples in Figure 9 correspond to the correlations between AIG and DIS, and between CVV and INTC, respectively. The vertical line separates the sample estimation period from the out of sample forecast period. The DCC forecasts show the slowest movement (see long dashed lines) since this model has the highest persistence. In contrast, the FSG-DCC forecasts (bold solid line) mean revert faster to the low frequency correlation forecast (solid line). The FG-DCC model appears as an intermediate case in terms of mean reversion speed and its long term forecast converges to the constant in Equation (19), which seems to approximate well the sample correlation. Furthermore, the first graph shows a case in which we might expect similar long horizon forecasts from the three models since the low frequency correlation at the end of the estimation period is flat and similar to the sample correlation. In contrast, the second graph presents an example in which long term forecasts from the FSG-DCC will differ considerably from those of the other two models due to a large discrepancy between the sample correlation and the trend in the low frequency correlation observed at the end of the estimation period. The second graph also illustrates that, contrary to the DCC model, the two factor correlation models might show non-monotonic mean reversion since their individual components might be associated with different long memory features.

We now proceed to evaluate the forecast performance of the three models mentioned above. The evaluation is conducted in-sample and out-of-sample. The first exercise considers the long run forecast performance of such models taking into account their in-sample forward looking properties. Specifically, given that the spline volatilities know ex-ante their future paths, the in-sample exercise rules out this foresight advantage by constructing long horizon correlation forecasts at each point in the sample keeping the spline functions fixed during the forecast period. This implies mean reversion of the long
run forecasts according to Equation (15). We consider a period of 100 days as our long run horizon.\textsuperscript{15} Regarding the out-of-sample forecast exercise, we consider multiple-step ahead forecasts assuming, as before, that spline volatilities remain constant during the forecast period at the values observed on the last day of the sample.\textsuperscript{16}

Our forecast comparison follows the approach of Engle and Colacito (2006) by using an economic loss function to assess the performance of each model. Different from them, however, we use forward hedging portfolios based on forecasts of the covariance matrix associated with each model, for a given horizon. Specifically, we focus on a portfolio problem where an investor wants to optimize today her forward asset allocation given a forward conditional covariance matrix. In the classical mean variance setup, this problem can be formulated as:

\[
\begin{align*}
\min_{w_{t+k}} & \quad w'_{t+k} H_{t+k} w_{t+k} \\
\text{s.t.} & \quad w'_{t+k} \mu = \mu_0,
\end{align*}
\]

where \(\mu\) is the vector of expected excess returns, \(w_{t+k}\) denotes a portfolio at time \(t+k\) that was formed using the information at time \(t\), and \(H_{t+k}\) is a \(k\) steps ahead forecast (made at time \(t\)) of the conditional covariance matrix of excess returns. So, the solution to (24) is:

\[
\begin{align*}
&w_{t+k} = \frac{H^{-1}_{t+k} \mu}{\mu' H^{-1}_{t+k} \mu} \mu_0,
\end{align*}
\]

and it represents optimal forward portfolio weights given the information at time \(t\).

Each covariance forecast \(H^{(j)}_{t+k}\) implies a particular forward portfolio \(w^{(j)}_{t+k}\) given a vector of expected returns. However, an important issue arises if we do not know the true vector of expected returns. Engle and Colacito (2006) point out that a direct comparison of optimal portfolio volatilities can be misleading when we use estimates of expected

\textsuperscript{15} Of course, there are many other possibilities to use for long term horizons. However, based on the daily frequency of our data, a horizon of 100 days can be seen as a reasonable lower bound.

\textsuperscript{16} To keep our exercise simple, we just took a time series approach here. Alternatively, according to our discussion in Section III, it is possible to use economic variables to forecast market low frequency volatilities following the approach of Engle and Rangel (2006). Other possibilities include forward evaluations of the spline functions or setting a distribution over possible paths.
returns to compute such volatilities. They propose a framework that isolates the effect of covariance information based on the asymptotic properties of sample standard deviations of optimized portfolio returns. We follow their approach and compare the standard deviations of returns on forward hedge portfolios formed with each model’s covariance matrix forecast. Specifically, for the in-sample exercise, at each point in the sample and for each model, we construct the corresponding long horizon (k steps ahead) covariance forecast; then we form optimized forward hedge portfolios using such forecast and Equation (25) along with selected vectors of expected returns associated with different hedges and a required return $\mu_0 = 1$. Thus, given a sample of size $T$, the in-sample standard deviations of returns on long term optimized forward hedge portfolios are given by:

$$\sigma_{p,ls}^{(j)} = \sqrt{\frac{1}{T-k} \sum_{t=1}^{T-k} \left( w_{p,t+k}^{(j)} (r_{t+k} - \bar{r}) \right)^2}, \quad j = FSG, FG - DCC, DCC, \quad p = 1, 2, ..., N,$$

(26)

where $\bar{r}$ denotes the sample mean of returns and $w_{p,t+k}^{(j)}$ corresponds to a portfolio associated with model $j$ where asset $p$ is hedged against the other assets. For the out-of-sample exercise, we construct multiple-step ahead forecasts from each model at time $T$; thus, given an out-of-sample period of $H$ days and the same required return, the standard deviations of returns on out-of-sample optimized forward hedge portfolio are:

$$\sigma_{p,os}^{(j)} = \sqrt{\frac{1}{H} \sum_{k=1}^{H} \left( w_{p,T+k}^{(j)} (r_{T+k} - \bar{r}) \right)^2}, \quad j = FSG, FG - DCC, DCC, \quad p = 1, 2, ..., N.$$

(27)

Table 6 presents the results for the in-sample long horizon forecast exercise. It is found that the FSG-DCC model outperforms the other models for the considered horizon. The FSG-DCC model obtains smaller volatilities for thirty cases whereas the FG-DCC is superior only for three cases. The standard DCC model is always inferior. Moreover, our results indicate that, on average, using long horizon FSG-DCC correlation forecasts to hedge reduces the standard deviation of the corresponding optimized portfolios on about

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A hedge is constructed holding one asset for return and the others for hedge. Thus, we have vectors of expected returns with one entry equal to one and everything else set to zero.
18 basis points, when it is compared with that of FG-DCC portfolios, and on about 78 basis points, when it is compared with that of DCC portfolios.

Our final exercise extends the evaluation to an out-of-sample period considering multiple forecast horizons. Since our estimation period goes from 12/01/1988 to 12/29/2006, we evaluate volatilities of optimized forward portfolio returns at horizons from 1 to 124 days ahead (from 01/02/2007 to 06/29/2007). For each model, Figure 10 presents the mean and median of square returns on the optimized forward hedge portfolios at multiple horizons. Forward hedge portfolios constructed using forecasts from the DCC model are associated with the largest mean square return. Portfolios constructed from forecasts based on the FSG-DCC or the FG-DCC models show the smallest mean square returns. Although these values are very close, the mean square return of FG-DCC forward hedge portfolios is slightly smaller. Similarly, in terms of median square returns, those associated with portfolios from the FSG-DCC or the FG-DCC models present the smallest values; however, the median of forward hedge portfolios from the FSG-DCC model is slightly smaller in this case.

Overall, there is evidence that the FSG-DCC model has important advantages for forecasting correlations at relatively long horizons. In addition, the two factor correlation models perform very well when we consider multiple horizons. Indeed, the FSG-DCC and its restricted version, the FG-DCC model, are good candidates for shorter horizons. Moreover, exploring different possibilities to construct FSG-DCC forecasts using economic variables leaves room for further improvement at both long and short horizons. We leave these extensions for future research.

VI Concluding Remarks

We develop a structural factor model for correlations that characterizes dynamic patterns at high and low frequencies. Exploiting the factor structure and the dynamic properties of low frequency market and idiosyncratic volatilities, we proxy the low frequency

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18 124 is the number of active trading days during the forecast period.
component of correlations through a smooth function of time and the corresponding factor loadings. Our semi-parametric approach generalizes dynamic conditional correlation models by allowing the high frequency correlation component to mean revert toward the time varying low frequency component.

At high frequencies, our structural model incorporates dynamic effects that arise from relaxing assumptions in the standard one-factor CAPM model. Such effects account for time varying betas and missing pricing factors.

We present a simple maximum likelihood procedure to estimate the high and low frequency components of the correlation structure. This procedure involves two simple steps: the first one involves estimation of betas, and Spline-GARCH high and low frequency volatility components of factor(s) and idiosyncratic returns; the second stage involves estimation of dynamic conditional correlations among factor(s) and idiosyncratic terms.

Our structural model successfully captures dynamic empirical features of the data in the US market and it facilitates linking economic variables to the structure of correlations. Moreover, we find that the model is competitive for forecasting applications. It shows important advantages for forecasting correlations at relatively long horizons. These advantages are related to its flexibility to adjust the level of mean reversion through time. On average, the factor correlation models presented in this study outperform the more parsimonious and simpler DCC model to forecast equity correlations at short and long horizons. Although this comes at a cost of estimating more parameters, the approach presented in this paper is easy to implement and the estimation process remains tractable even if we include a large number of assets.

Finally, our empirical analysis also evaluates the relevance of the assumptions in a range of one-factor CAPM models based on the ability of the resulting correlation structures to match empirical features of the data. Our results suggest that, besides the importance of
modeling market volatility behavior, incorporating a richer dynamics in the idiosyncratic terms improves the empirical performance of this class of models.

References


Appendix A1

Proof of Proposition 1: Consider a single factor model with time varying betas satisfying assumptions i)-v). The equation for returns (in excess of the risk free rate) can be written as:

\[ r_{it} = \beta_{it} r_{mt} + \tilde{u}_{it}, \quad (28) \]

where \( \beta_{it} = \beta_i + w_{it} \). If we define \( u_{it} \equiv w_{it} r_{mt} + \tilde{u}_{it}, \forall i = 1, ..., N \), then we can rewrite (28) as:

\[ r_{it} = \beta_i r_{mt} + u_{it}. \quad (29) \]

Note that assumption iv) along with the Law of Iterated Expectations (LIE) imply

\[ E(w_{it} r_{mt}) = E(E_{t-1}(w_{it} r_{mt})) = E(E_{t-1}(w_{it})E_{t-1}(r_{mt})) = 0 \]

since \( E_{t-1}(r_{mt}) = 0 \) by the no-arbitrage condition. Thus, \( E(u_{it}) = E(w_{it} r_{mt}) + E(\tilde{u}_{it}) = 0, \forall i \). However, the new errors will be conditionally correlated with the market factor since, by iv) and the LIE,

\[ E_{t-1}(w_{it}^2 r_{mt}^2) = E_{t-1}(w_{it}) Var_{t-1}(r_{mt}), \quad (30) \]

which is in general different from zero. Thus,

\[ E_{t-1}(r_{jit}) = \beta_i \beta_j E_{t-1}(r_{mt}^2) + \beta_i E_{t-1}(r_{mt} u_{jt}) + \beta_j E_{t-1}(r_{mt} u_{it}) + E_{t-1}(u_{it} u_{jt}), \quad (31) \]

and

\[ E_{t-1}(r_{ki}) = \beta_k E_{t-1}(r_{mt}^2) + E_{t-1}(u_{kt}^2) + 2 \beta_k E_{t-1}(r_{mt} u_{kt}), \quad k = i, j. \quad (32) \]

Using again the LIE and ii), we have:

\[ E_{t-1}(w_{it} w_{jt} r_{mt}) = Var_{t-1}(r_{mt}) E_{t-1}(w_{it} w_{jt}) = 0, \forall i \neq j. \quad (33) \]

Also, from iii)

\[ E_{t-1}(w_{jt} r_{mt} \tilde{u}_{it}) = E_{t-1}(w_{jt} r_{mt} \tilde{u}_{jt}) = E_{t-1}(\tilde{u}_{jt} \tilde{u}_{it}) = 0, \forall i \neq j. \quad (34) \]

Therefore, \( E_{t-1}(u_{it} u_{jt}) = 0, \forall i \neq j \) and the conditional correlation between asset i and j is given by Equation (8):

\[ \rho_{i,j} = \frac{\beta_i \beta_j E_{t-1}(r_{mt}^2) + \beta_i E_{t-1}(r_{mt} u_{jt}) + \beta_j E_{t-1}(r_{mt} u_{it})}{\sqrt{\beta_i^2 E_{t-1}(r_{mt}^2) + E_{t-1}(u_{it})^2 + 2 \beta_i E_{t-1}(r_{mt} u_{it}) \sqrt{\beta_j^2 E_{t-1}(r_{mt}^2) + E_{t-1}(u_{jt})^2 + 2 \beta_j E_{t-1}(r_{mt} u_{jt})}}} . \]

When we allow \( E_{t-1}(u_{it} u_{jt}) \neq 0 \) by relaxing ii) and iii), Equation (9) follows. ■
Appendix A2

Proof of Proposition 2: Consider the following vectors of returns, factor loadings, and innovations: \( r_t = (r_{t1}, r_{t2}, \ldots, r_{TN})' \), \( \beta = (\beta_{t1}, \beta_{t2}, \ldots, \beta_{TN})' \), and \( u_t = (u_{t1}, u_{t2}, \ldots, u_{TN})' \). Given a vector \( F_t \) of common factor(s), we can rewrite the model in Equation (1) as:

\[
r_t = \beta F_t + u_t.
\]

Thus, given the \( t-1 \) information set \( \mathcal{I}_{t-1} \), the conditional covariance matrix is:

\[
E_{t-1}(r_{t} r_{t}') = \beta E_{t-1}(F_{t} F_{t}') \beta' + \beta E_{t-1}(F_{t} u_{t}') + E_{t-1}(u_{t} F_{t}') \beta' + E_{t-1}(u_{t} u_{t}').
\]

In particular, for the one-factor CAPM case, \( F_t = r_{mt} \) and (36) takes the following form:

\[
E_{t-1}(r_{t} r_{t}') = E_{t-1}(r_{mt} r_{mt}') \beta \beta' + \beta E_{t-1}(r_{mt} u_{t}') + E_{t-1}(u_{t} r_{mt}') \beta' + E_{t-1}(u_{t} u_{t}').
\]

From equations (9) and (10), the typical \((i,j)\) element of the first term on the RHS of (37) is:

\[
\beta_i \beta_j E_{t-1}(r_{mt}^2) = \beta_i \beta_j \tau_{mi,mj} g_{mt}.
\]

Similarly, from Equations (10), (11), and (12), the typical \((i,j)\) element of the second term is:

\[
\beta_i E_{t-1}(r_{mt} u_{jt}) = \beta_i \sqrt{\tau_{mi,mj} g_{mt}} \sqrt{\tau_{jt,ij} g_{jt} \rho^\tau_{m,j,t}},
\]

the typical \((i,j)\) element of the third term is:

\[
\beta_j E_{t-1}(r_{mt} u_{it}) = \beta_j \sqrt{\tau_{mi,mj} g_{mt}} \sqrt{\tau_{it,ij} g_{it} \rho^\tau_{m,j,t}},
\]

and the typical \((i,j)\) element of the last term is:

\[
E_{t-1}(u_{it} u_{jt}) = \sqrt{\tau_{it,ij} g_{it}} \sqrt{\tau_{jt,ij} g_{jt} \rho^\tau_{i,j,t}}.
\]

Equation (13) follows from substituting these conditional expectations into Equation (9).

The low frequency correlation is derived from the unconditional version of Equation (9):

\[
\bar{\rho}_{ij} = \frac{\beta_i \beta_j E(r_{mt}^2) + \beta_i E(r_{mt} u_{jt}) + \beta_j E(r_{mt} u_{it}) + E(u_{it} u_{jt})}{\sqrt{\beta_i^2 E(r_{mt}^2) + E(u_{it})^2 + 2 \beta_i E(r_{mt} u_{it}) \sqrt{\beta_j^2 E(r_{mt}^2) + E(u_{jt})^2 + 2 \beta_j E(r_{mt} u_{jt})}}},
\]

Under the assumption that the factor(s) and the idiosyncrasies are unconditionally uncorrelated, we have:
\[
\tilde{\rho}_{i,j} = \frac{\beta_i \beta_j E(r_{m}^2) + E(u_i u_j)}{\sqrt{\beta_i^2 E(r_{m}^2) + E(u_i^2)} \sqrt{\beta_j^2 E(r_{m}^2) + E(u_j^2)}}. \tag{42}
\]

Now, from (10), (11), and the LIE:
\[
E(r_{m}) = \tau_m E(g_{m}) = \tau_m,
\]
and
\[
E(u_i^2) = \tau_i E(g_i) = \tau_i, \quad \forall i = 1, 2, \ldots, N.
\]

Also,
\[
E(u_i u_j) = \frac{E(g_i^{1/2} e_u g_j^{1/2} e_u)}{\sqrt{\tau_i} \sqrt{\tau_j}} = \text{corr}(g_i^{1/2} e_u, g_j^{1/2} e_u) \equiv \tilde{\rho}_{i,j}.
\]

Note that \(\rho^{\varepsilon}_{i,j,t} = \rho^{\varepsilon}_{i,j,t}, \quad \forall t\), thus we approximate \(\tilde{\rho}_{i,j}\) with the sample correlation, \(\tilde{\rho}_{i,j}\), from Equation (11). Plugging in the previous expressions into (42), we obtain the time varying low frequency correlation in Equation (14).

Moreover, if we assume \(E_i(\tau_{k,t+h}) = \tau_{k,t}, \forall h > 0, k = 1, 2, \ldots, N\), then the long horizon forecast of (13) can be constructed using the mean reversion properties of the GARCH and DCC equations. Indeed, the GARCH dynamics implies \(\lim_{h \to \infty} g_{k,t+h} = 1, \forall k = m, 1, 2, \ldots, N\). Also, the long horizon correlation forecasts associated with the vector of innovations are given by the terms targeting correlations (see Equation (12)):
\[
\lim_{h \to \infty} \rho^{\varepsilon}_{i,j,t+h} = \lim_{h \to \infty} \frac{q_{i,j,t+h}}{\sqrt{q_{i,j,t+h}} \sqrt{q_{i,j,t+h}}} = \frac{\tilde{\rho}_{i,j}^\varepsilon}{\sqrt{\tilde{\rho}_{i,j}^ \varepsilon \sqrt{\tilde{\rho}_{j,i}^ \varepsilon}}} = \rho^{\varepsilon}_{i,j}, \tag{43}
\]
\(\forall i, j \in \{1, 2, \ldots, N\}\). In addition, from our assumption that the idiosyncrasies are unconditionally uncorrelated with the factor, \(\tilde{\rho}_{m,i} = 0\) and \(\lim_{h \to \infty} \rho_{m,i,t+h} = 0, \forall i = 1, \ldots, N\).

Hence, substituting the long run forecasts of each term into (13), we obtain:
\[
\lim_{h \to \infty} \tilde{\rho}_{i,j,t+h} = \frac{\beta_i \beta_j \tau_m + \sqrt{\tau_i \sqrt{\tau_j \tilde{\rho}_{i,j}^\varepsilon}}}{\sqrt{\beta_i^2 \tau_m + \tau_i \sqrt{\beta_j^2 \tau_m + \tau_j}}}, \tag{44}
\]
which coincides with the low frequency correlation.■
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<th>Annualized Sample Volatility</th>
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</tr>
<tr>
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<td>CAT</td>
<td>0.06%</td>
<td>0.49</td>
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<td>Exxon Mobil</td>
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<td>0.25</td>
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Source: Daily returns from CRSP. The annualized sample volatility is the square root of the sample average of annualized daily squared returns.
Table 2


\[ r_{it} = \beta_i r_{mt} + u_{it}, \quad r_{mt} = \sqrt{\tau_{it}} g_{mt} \varepsilon_t, \quad u_{it} = \sqrt{\tau_{it}} g_{it} \varepsilon_{it}, \quad \text{where} \quad \varepsilon_{it} \mid \Phi_{t-1} \sim N(0,1), \forall i = m, 1, 2, ..., N \]

\[ g_{mt} = (1 - a_m - b_m) + a_m \left( \frac{r_{mt-1}^2}{\tau_{mt-1}} \right) + b_m g_{mt-1}, \quad g_{it} = (1 - a_i - b_i) + a_i \left( \frac{(r_{it-1} - \beta_i r_{mt-1})^2}{\tau_{it-1}} \right) + b_i g_{it-1}, \forall i = m, 1, ..., N \]

\[ \tau_{it} = c_i \exp \left( w_{it} t + \sum_{r=1}^{k} w_{ir} \left( t - t_{i-1} \right)_+ \right), \quad \forall i = m, 1, ..., N \]

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<th>( a_i )</th>
<th>T-Stat</th>
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</table>
\[ (\varepsilon_{i1}^m, \varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{iN})' \sim \text{DCC}, \]

\[ \rho_{i,j}^\varepsilon = \frac{q_{i,j}^\varepsilon}{\sqrt{q_{i,j}^\varepsilon \sqrt{q_{i,j}^\varepsilon}}}, \]

\[ q_{i,j} = \bar{\rho}_{i,j}^\varepsilon + a_{\text{DCC}}(\varepsilon_{i,j-1}\varepsilon_{j-1} - \bar{\rho}_{i,j}^\varepsilon) + b_{\text{DCC}}(q_{i,j-1} - \bar{\rho}_{i,j}^\varepsilon), \quad \forall i, j \in \{1, \ldots, N\}, \]

\[ q_{m,i} = \bar{\rho}_{m,i}^\varepsilon + a_{\text{DCC}}(\varepsilon_{i,j-1}\varepsilon_{j-1} - \bar{\rho}_{m,i}^\varepsilon) + b_{\text{DCC}}(q_{m,j-1} - \bar{\rho}_{m,i}^\varepsilon), \quad \forall i \in \{m, 1, \ldots, N\} \]

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<td>(b_{\text{DCC}})</td>
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a) The risk free rate term has been dropped from the returns equation to simplify notation.

b) The optimal number of knots was selected based on the BIC.
### Table 3

**Average Annualized Idiosyncratic Volatilities vs Average Correlations**

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<th></th>
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</thead>
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<td>0.2643</td>
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<tr>
<td>JPM</td>
<td>0.2547</td>
<td>0.2941</td>
<td>0.3102</td>
</tr>
<tr>
<td>KO</td>
<td>0.2038</td>
<td>0.2888</td>
<td>0.2794</td>
</tr>
<tr>
<td>MCD</td>
<td>0.2349</td>
<td>0.2409</td>
<td>0.2390</td>
</tr>
<tr>
<td>MMM</td>
<td>0.1885</td>
<td>0.2983</td>
<td>0.2981</td>
</tr>
<tr>
<td>MO</td>
<td>0.2681</td>
<td>0.2262</td>
<td>0.2176</td>
</tr>
<tr>
<td>MRK</td>
<td>0.2338</td>
<td>0.2724</td>
<td>0.2582</td>
</tr>
<tr>
<td>MSFT</td>
<td>0.2785</td>
<td>0.2413</td>
<td>0.2708</td>
</tr>
<tr>
<td>PFE</td>
<td>0.2534</td>
<td>0.2477</td>
<td>0.2568</td>
</tr>
<tr>
<td>PG</td>
<td>0.2059</td>
<td>0.2698</td>
<td>0.2705</td>
</tr>
<tr>
<td>T</td>
<td>0.2261</td>
<td>0.2541</td>
<td>0.2552</td>
</tr>
<tr>
<td>UTX</td>
<td>0.2194</td>
<td>0.2971</td>
<td>0.2918</td>
</tr>
<tr>
<td>VZ</td>
<td>0.2210</td>
<td>0.2538</td>
<td>0.2495</td>
</tr>
<tr>
<td>WMT</td>
<td>0.2289</td>
<td>0.2921</td>
<td>0.2962</td>
</tr>
<tr>
<td>XOM</td>
<td>0.1885</td>
<td>0.2596</td>
<td>0.2458</td>
</tr>
</tbody>
</table>

a) For correlations, the average is taken over the correlations of each asset with respect to all the others.

b) Average Annualized Idiosyncratic Volatility of asset \( i = \frac{1}{T} \sum_{t=1}^{T} \tau_{it}^{1/2} \), where \( \tau_{it}^{1/2} \) is the annualized low frequency volatility for asset \( i \) at time \( t \).

c) Avg. low freq. correlation of asset \( i = \frac{1}{T (N-1)} \sum_{j \neq i}^{T} \sum_{t=1}^{T} \bar{\rho}_{i,j,t}, \bar{\rho}_{i,j,t} = \frac{\beta_i \beta_j \tau_{mt}}{\sqrt{\beta_i^2 \tau_{mt}^2 + \tau_{it}^2 \bar{\rho}_{i,j}^z}} \sqrt{\beta_j^2 \tau_{mt} + \tau_{it}^2 \bar{\rho}_{i,j}^z + \tau_{jt}^2} \).

d) For average rolling correlations, we replace \( \bar{\rho}_{i,j,t} \) in c) by the corresponding rolling correlations.
Table 4
Correlation Models from Factor Assumptions

<table>
<thead>
<tr>
<th>Low Frequency</th>
<th>Factor Component</th>
<th>High Frequency</th>
<th>Factor and Idiosyncratic Components</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PANEL 1: DYNAMIC VOLATILITY COMPONENTS</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FC-C: Constant factor and idiosyncratic volatilities</td>
<td>FG-C: GARCH factor volatility</td>
<td>FG-G: GARCH factor and Idiosyncratic volatilities</td>
<td></td>
</tr>
<tr>
<td>ρ_{i,j}^{FC-C} = \frac{\beta_i \beta_j \sigma_m^2}{\sqrt{\beta_i^2 \sigma_m^2 + \sigma_i^2 \sqrt{\beta_i^2 \sigma_m^2 + \sigma_j^2}}}</td>
<td>ρ_{i,j}^{FG-C} = \frac{\beta_i \beta_j h_{mt}}{\sqrt{\beta_i^2 h_{mt} + \sigma_i^2 \sqrt{\beta_i^2 h_{mt} + \sigma_j^2}}}</td>
<td>ρ_{i,j}^{FG-G} = \frac{\beta_i \beta_j h_{mt} + h_{it} \sqrt{\beta_i^2 h_{mt} + h_{it}}}{\sqrt{\beta_i^2 h_{mt} + h_{it} \sqrt{\beta_i^2 h_{mt} + h_{it}}}}</td>
<td></td>
</tr>
<tr>
<td>FS-S: Low frequency spline factor and idiosyncratic volatilities</td>
<td>FSG-C: Spline-GARCH factor volatility</td>
<td>FSG-SG: Spline-GARCH factor and Idiosyncratic volatilities</td>
<td></td>
</tr>
<tr>
<td>ρ_{i,j}^{FS-S} = \frac{\beta_i \beta_j \tau_{mt}}{\sqrt{\beta_i^2 \tau_{mt} + \tau_{it} \sqrt{\beta_i^2 \tau_{mt} + \tau_{jt}}}}</td>
<td>ρ_{i,j}^{FSG-C} = \frac{\beta_i \beta_j \tau_{mt} g_{mt}}{\sqrt{\beta_i^2 \tau_{mt} g_{mt} + \sigma_i^2 \sqrt{\beta_i^2 \tau_{mt} g_{mt} + \sigma_j^2}}}</td>
<td>ρ_{i,j}^{FSG-SG} = \frac{\beta_i \beta_j \tau_{mt} g_{mt} + \tau_{it} g_{it} \sqrt{\beta_i^2 \tau_{mt} g_{mt} + \tau_{it} g_{it}}}{\sqrt{\beta_i^2 \tau_{mt} g_{mt} + \tau_{it} g_{it} \sqrt{\beta_i^2 \tau_{mt} g_{mt} + \tau_{it} g_{it}}}}</td>
<td></td>
</tr>
</tbody>
</table>

**PANEL 2: OTHER DYNAMIC COMPONENTS**

<table>
<thead>
<tr>
<th>Betas</th>
<th>Idiosyncratic Correlations (Latent Factors)</th>
<th>All Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>FG-BDCC: FG-G with time varying betas</td>
<td>FG-IDCC: FG-G with latent factors</td>
<td>FG-DCC</td>
</tr>
<tr>
<td>ρ_{i,j,t}^{(4.1)} = \frac{\beta_i \beta_j h_{mt} + C}{\sqrt{A_1 \sqrt{B_1}}}</td>
<td>ρ_{i,j,t}^{(5.1)} = \frac{\beta_i \beta_j h_{mt} + h_{it} \sqrt{h_{it} \rho_{i,j}^{u}}}{{\sqrt{\beta_i^2 h_{mt} + h_{it} \sqrt{\beta_i^2 h_{mt} + h_{it}}}}}</td>
<td>See Equation (18)</td>
</tr>
<tr>
<td>FG-SG: FSG-G with time varying betas</td>
<td>FG-IDCC: FSG-SG with latent factors</td>
<td>FSG-DCC</td>
</tr>
<tr>
<td>ρ_{i,j,t}^{(4.2)} = \frac{\beta_i \beta_j \tau_{mt} g_{mt} + C}{\sqrt{A_2 \sqrt{B_2}}}</td>
<td>ρ_{i,j,t}^{(5.2)} = \frac{\beta_i \beta_j \tau_{mt} g_{mt} + \tau_{it} g_{it} \sqrt{\tau_{it} g_{it} \rho_{i,j}^{u}}}{\sqrt{\beta_i^2 \tau_{mt} g_{mt} + \tau_{it} g_{it} \sqrt{\beta_i^2 \tau_{mt} g_{mt} + \tau_{it} g_{it}}}}</td>
<td>See Equation (13)</td>
</tr>
</tbody>
</table>

a) σ denotes constant (C) volatilities, h and g refer to GARCH (G) and Spline-GARCH (SG) variances, respectively. b) These models are parametrizations of (8) and (9) in Prop 1.
Table 5
Evaluation of Factor Correlation Models

<table>
<thead>
<tr>
<th>Assumption</th>
<th>Model</th>
<th>Log Likelihood</th>
<th>Parameters</th>
<th>Likelihood Ratio</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>FC-C</td>
<td>-426550</td>
<td>67</td>
<td>45980*</td>
<td>-186.96</td>
</tr>
<tr>
<td></td>
<td>FS-S</td>
<td>-439690</td>
<td>315</td>
<td>19700*</td>
<td>-192.26</td>
</tr>
<tr>
<td></td>
<td>FG-C</td>
<td>-427110</td>
<td>69</td>
<td>44860*</td>
<td>-187.20</td>
</tr>
<tr>
<td>2</td>
<td>FSG-C</td>
<td>-427160</td>
<td>75</td>
<td>44760*</td>
<td>-187.21</td>
</tr>
<tr>
<td></td>
<td>FG-G</td>
<td>-440670</td>
<td>135</td>
<td>17740*</td>
<td>-193.03</td>
</tr>
<tr>
<td>3</td>
<td>FSG-SG</td>
<td>-442820</td>
<td>383</td>
<td>13440*</td>
<td>-193.51</td>
</tr>
<tr>
<td></td>
<td>FG-BDCC</td>
<td>-441170</td>
<td>168</td>
<td>16740*</td>
<td>-193.19</td>
</tr>
<tr>
<td>4</td>
<td>FSG-BDCC</td>
<td>-443340</td>
<td>416</td>
<td>12400*</td>
<td>-193.68</td>
</tr>
<tr>
<td></td>
<td>FG-IDCC</td>
<td>-446930</td>
<td>663</td>
<td>5220*</td>
<td>-194.80</td>
</tr>
<tr>
<td></td>
<td>FG-IDCC</td>
<td>-449100</td>
<td>911</td>
<td>880*</td>
<td>-195.29</td>
</tr>
<tr>
<td></td>
<td>FG-DCC</td>
<td>-447330</td>
<td>696</td>
<td>4420*</td>
<td>-194.91</td>
</tr>
<tr>
<td>All</td>
<td>FSG-DCC</td>
<td>-449540</td>
<td>944</td>
<td></td>
<td>-195.42</td>
</tr>
</tbody>
</table>

a) This column corresponds to the assumptions that are weakened in the factor specifications:
   1. Constant low frequency volatilities.
   2. Constant high frequency volatility of the market factor.
   3. Constant high frequency volatility of the idiosyncratic component of returns.
   4. Constant betas.
   5. One single common factor.
   The models are described in Table 4.

b) The last step is associated with the full FG-DCC and FSG-DCC models.

c) The likelihood ratio test compares the models on each row with the FSG-DCC model in the last row. *) Indicates that the restricted model is rejected at the 1% level.
### Table 6

<table>
<thead>
<tr>
<th>Forward Hedge Portfolios</th>
<th>FSG-DCC</th>
<th>FG-DCC</th>
<th>DCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{AA}$</td>
<td>0.2565</td>
<td>0.2598</td>
<td>0.2575</td>
</tr>
<tr>
<td>$\mu_{AIG}$</td>
<td>0.2032</td>
<td>0.2055</td>
<td>0.2315</td>
</tr>
<tr>
<td>$\mu_{AXP}$</td>
<td>0.2412</td>
<td>0.2442</td>
<td>0.2451</td>
</tr>
<tr>
<td>$\mu_{BA}$</td>
<td>0.2625</td>
<td>0.2640</td>
<td>0.2660</td>
</tr>
<tr>
<td>$\mu_{C}$</td>
<td>0.2386</td>
<td>0.2406</td>
<td>0.2596</td>
</tr>
<tr>
<td>$\mu_{CAT}$</td>
<td>0.2525</td>
<td>0.2551</td>
<td>0.2544</td>
</tr>
<tr>
<td>$\mu_{CVX}$</td>
<td>0.1673</td>
<td>0.1630</td>
<td>0.1696</td>
</tr>
<tr>
<td>$\mu_{DD}$</td>
<td>0.2102</td>
<td>0.2128</td>
<td>0.2132</td>
</tr>
<tr>
<td>$\mu_{DIS}$</td>
<td>0.2678</td>
<td>0.2667</td>
<td>0.2699</td>
</tr>
<tr>
<td>$\mu_{GE}$</td>
<td>0.1772</td>
<td>0.1805</td>
<td>0.1863</td>
</tr>
<tr>
<td>$\mu_{GM}$</td>
<td>0.2800</td>
<td>0.2826</td>
<td>0.3035</td>
</tr>
<tr>
<td>$\mu_{GT}$</td>
<td>0.3357</td>
<td>0.3384</td>
<td>0.3400</td>
</tr>
<tr>
<td>$\mu_{HD}$</td>
<td>0.2593</td>
<td>0.2633</td>
<td>0.2631</td>
</tr>
<tr>
<td>$\mu_{HON}$</td>
<td>0.2702</td>
<td>0.2721</td>
<td>0.2995</td>
</tr>
<tr>
<td>$\mu_{HPQ}$</td>
<td>0.3408</td>
<td>0.3417</td>
<td>0.3467</td>
</tr>
<tr>
<td>$\mu_{IBM}$</td>
<td>0.2553</td>
<td>0.2594</td>
<td>0.2574</td>
</tr>
<tr>
<td>$\mu_{NFC}$</td>
<td>0.3297</td>
<td>0.3321</td>
<td>0.3354</td>
</tr>
<tr>
<td>$\mu_{IP}$</td>
<td>0.2269</td>
<td>0.2280</td>
<td>0.2274</td>
</tr>
<tr>
<td>$\mu_{JNJ}$</td>
<td>0.1890</td>
<td>0.1900</td>
<td>0.1941</td>
</tr>
<tr>
<td>$\mu_{JPM}$</td>
<td>0.2598</td>
<td>0.2640</td>
<td>0.2639</td>
</tr>
<tr>
<td>$\mu_{KO}$</td>
<td>0.2048</td>
<td>0.2059</td>
<td>0.2081</td>
</tr>
<tr>
<td>$\mu_{MCD}$</td>
<td>0.2420</td>
<td>0.2427</td>
<td>0.2464</td>
</tr>
<tr>
<td>$\mu_{MMM}$</td>
<td>0.1868</td>
<td>0.1882</td>
<td>0.1911</td>
</tr>
<tr>
<td>$\mu_{MO}$</td>
<td>0.2779</td>
<td>0.2808</td>
<td>0.2869</td>
</tr>
<tr>
<td>$\mu_{MRK}$</td>
<td>0.2163</td>
<td>0.2187</td>
<td>0.2199</td>
</tr>
<tr>
<td>$\mu_{MSFT}$</td>
<td>0.2734</td>
<td>0.2735</td>
<td>0.3014</td>
</tr>
<tr>
<td>$\mu_{PFE}$</td>
<td>0.2279</td>
<td>0.2291</td>
<td>0.2303</td>
</tr>
<tr>
<td>$\mu_{PG}$</td>
<td>0.2100</td>
<td>0.2124</td>
<td>0.2124</td>
</tr>
<tr>
<td>$\mu_{T}$</td>
<td>0.2104</td>
<td>0.2125</td>
<td>0.2128</td>
</tr>
<tr>
<td>$\mu_{TX}$</td>
<td>0.2170</td>
<td>0.2184</td>
<td>0.2418</td>
</tr>
<tr>
<td>$\mu_{VZ}$</td>
<td>0.2054</td>
<td>0.2069</td>
<td>0.2056</td>
</tr>
<tr>
<td>$\mu_{WMT}$</td>
<td>0.2338</td>
<td>0.2344</td>
<td>0.2351</td>
</tr>
<tr>
<td>$\mu_{XOM}$</td>
<td>0.1554</td>
<td>0.1552</td>
<td>0.1601</td>
</tr>
<tr>
<td><strong>All Portfolios</strong></td>
<td>0.2430</td>
<td>0.2448</td>
<td>0.2508</td>
</tr>
</tbody>
</table>

Notes: Sample standard deviations of returns on optimized forward hedge portfolios constructed at each point in the sample using 100 days ahead covariance forecasts from FSG-DCC, FG-DCC, and DCC models, respectively, and subject to a required return of 1. The stock in the corresponding row is hedged against all other stocks.
Notes: Beta estimated coefficients from the first estimation stage of the FSG-DCC model. See Equations (1) and (11).
Figure 2
Idiosyncratic Volatilities: High and Low Frequency Components

Notes: HFV=High frequency idiosyncratic volatility (See second equation in (11)). LFV=Low frequency idiosyncratic volatility (See third equation in (11)).
Figure 3
High and Low Frequency Market Volatility

Notes: HFVOL=High frequency market volatility (See second equation in (10)). LFVOL=Low frequency market volatility (See third equation in (10)).
Figure 4

Correlations for Stocks with Increasing Low frequency Idiosyncratic Volatility

- CVX-JPM
- CVX-T
- CVX-DIS

Legend:
- Blue line: Low Frequency Correlation
- Red line: High Frequency Correlation
- Dashed line: Rolling Correlation
Figure 5

Correlations for Stocks with Decreasing Low frequency Idiosyncratic Volatility
Figure 6

Average Idiosyncratic Volatilities vs Low Frequency Correlations

Average Idiosyncratic Volatilities vs Rolling Correlations

Notes: The average values are taken from Table 3. The stocks are sorted by low frequency volatility.
Figure 7

Average Low Frequency Market and Idiosyncratic Volatilities Over Two-Year Periods

Notes: The average low frequency volatility over the period $p$ is defined as follows:

Average Annualized Idiosyncratic Volatility of asset $i$ = $\frac{1}{T_p} \sum_{t=1}^{T_p} \tau_{it}^{1/2}$, where $\tau_{it}^{1/2}$ is the annualized low frequency volatility for asset $i$ at time $t$, $\forall i = m,1,2,...,N$, and $T_p$ is the number of daily observations in period $p$. 

45
Figure 8

Average Correlations

Notes: The average rolling correlation over the period $p$ is defined as follows:

$$
\rho_{ij}^{\text{Rolling}} = \frac{1}{N} \sum_{t=1}^{T_p} \left\{ \frac{1}{T_p (N-1)} \sum_{j \neq i} \sum_{t=1}^{T_p} \rho_{ij,t}^{\text{Rolling}} \right\},
$$

where $T_p$ denotes the number of daily observations in period $p$, and $\rho_{ij}^{\text{Rolling}}$ is the rolling correlation between assets $i$ and $j$ at time $t$. Similarly, average FSG-DCC correlations are constructed by replacing $\rho_{ij,t}^{\text{Rolling}}$ in the above expression by $\rho_{ij,t}$ (from Proposition 2).
Figure 9

Multiple-Step Ahead Forecasts
AIG and DIS

![Multiple-Step Ahead Forecasts Graph](image)

CVX and INTC

![CVX and INTC Graph](image)

Notes: HFC=High frequency correlation, LFC=Low frequency correlation. Forecast horizon: 130 days.
Notes: Mean and median over $p$ and $k$ of $\left\{ w_{p,T+k|j} \left( r_{T+k} - \bar{r} \right) \right\}^2$, where $\bar{r}$ denotes the sample mean of returns and $w_{p,T+k|j}$ denotes the optimal forward hedge portfolio weights associated with model $j$ and asset $p$. The portfolios are constructed using information up to time $T$, the last day in the sample (12/31/2006), and $k$ goes from 1 to 124 days ahead.