7. Introduction to the regression model

7.1 An example: Housing Data
7.2 Prediction in the regression model
7.3 Another Example
7.4 Fitted Values and Residuals
7.5 The Least Squares Criterion
7.6 R-squared

7.1 An example: Housing Data

Problem:

Predict market price of a house from observed characteristics

Solution:

Collect data on prices and observed characteristics and build a prediction rule that predicts price as a function of observed characteristics
Which characteristics should we use?

Size, # of bathrooms, location, age, condition, b-ball hoop in driveway, how badly owner wants to sell, etc.

Some of these are easy to quantify; others are not so easy...

For simplicity, we focus on size.

Goal: predict market price from size

The first step is always defining the variables.

X = size of house (in 1000’s of square feet)
Y = price of house (in 1000’s of dollars)

Dependent variable (denoted by Y):
quantity we seek to explain/predict

Independent variable (denoted by X):
quantity used to explain/predict the dependent variable
We collect data on several houses.

For a given house, an observation consists of the pair

\[(X_i, Y_i)\]

where

\[X_i = \text{size of the } i\text{-th house}\]
\[Y_i = \text{price of the } i\text{-th house}\]

---

The Data (housepr.xls)

<table>
<thead>
<tr>
<th>Size</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
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<tr>
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<td>3.20</td>
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</tr>
<tr>
<td>3.50</td>
<td>172</td>
</tr>
</tbody>
</table>

Our sample consists of 15 observations.
A “Scatter-Plot” of the Housing Data

But how would you *predict* the sales price of a house that is not in the data set?
So regression is fitting a line which amounts to finding intercept and slope parameters.

\[ Y = b_0 + b_1 X \]

- \( b_0 \) is the intercept
- \( b_1 \) is the slope

### 7.2 Prediction

**Regression** chooses a line that fits the linear pattern of the data.

Regression computes the regression line (values for \( b_0 \) and \( b_1 \) that can be used for prediction).
Output from regression

Results of simple regression for Price

Summary measures
- Multiple R: 0.9092
- R-Square: 0.8267
- StdErr of Est: 14.1384

ANOVA table
- Source: df | SS | MS | F | p-value
  - Explained: 1 | 12393.1077 | 12393.1077 | 61.9983 | 0.0000
  - Unexplained: 13 | 2598.6256 | 199.8943 |

Regression coefficients
- Coefficient | Std Err | t-value | p-value | Lower limit | Upper limit
  - Constant: 38.8847 | 9.0939 | 4.2759 | 0.0009 | 19.2385 | 58.5309
  - Size: 35.3860 | 4.4941 | 7.8739 | 0.0000 | 25.6771 | 45.0948

Plot the predicted outcomes

Pred = 38.9 + 35.4Size
Some unanswered questions:

- How does the software pick the line?
- How accurate is the prediction?
- What is all that other stuff in the output?

7.3 A second important interpretation of the regression line.

- The slope of the regression line tells us how much we change the predicted value as we change x.
- For the housing example this says that each additional 1000 square feet translates into an additional $35,386 in the predicted sales price!

(you can see why it is important to remember the units that we use to measure x and y)
7.4 Fitted Values and Residuals

We have data:

\((X_i, Y_i)\) for \(i = 1, \ldots, n\)

We have a line:

\(b_0, b_1\)

For the \(i\)-th observation, the **fitted value** is defined as

\[
\hat{Y}_i = b_0 + b_1 X_i
\]

The fitted value is “read off” the regression line:
The fitted value is “read off” the regression line:

For the i-th observation, the **residual** is defined as

\[ e_i = Y_i - \hat{Y}_i \]

Positive residual: fitted value “too low”

Negative residual: fitted value “too high”
The residual is the vertical distance between the true value and the regression line:

$$e_i = Y_i - \hat{Y}_i$$
Notice that

\[ e_i = Y_i - \hat{Y}_i \]

or, equivalently, after rearranging terms:

\[ Y_i = \hat{Y}_i + e_i \]

We see that we can use the fitted values and the residuals to decompose \( Y_i \) into two parts.

Important FACT: \( X_i \) and \( e_i \) are guaranteed to be uncorrelated! So are \( \hat{Y}_i \) and \( e_i \)!
7.5 The Least Squares Criterion

What we want:
- fitted values close to true values
or equivalently:
- residuals close to zero

Ideal world:
line passes through all the points
(residuals all zero)

Real world:
try to make all residuals “small”

For any chosen line, we want some **criterion**
that measures the size of *all* the residuals.

Then, pick the line (i.e., values of $b_0$ and $b_1$)
that minimizes the criterion!

*What criterion should we use?*
The most popular criterion for fitting a line is called **least squares**. This method says to choose $b_0$ and $b_1$ to minimize

$$
\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} e_i^2
$$

The sum of squares of the residuals

**Least-squares chooses the line** $(b_0$ and $b_1$) **so that the sum of the squared residuals is minimal!**

**Recall that** $\hat{Y}_i = b_0 + b_1 X_i$ **so** $b_0$ **and** $b_1$ **are in the fitted value.**

The formulae for $b_0$ and $b_1$ that minimize the least squares criterion are:

\[
b_1 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n} (X_i - \overline{X})^2}
\]

\[
b_0 = \overline{Y} - b_1 \overline{X}
\]

where the bars denote means:

\[
\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \overline{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i
\]
7.6 **R-squared**, how well does X predict Y?

We have:

\[ Y = \hat{Y} + e \]

This says that we have a data set Y that is the sum of two data sets, \( \hat{Y} \) and the error e.

Recall that: \( \text{Var}(Y) = \text{Var}(\hat{Y}) + \text{Var}(e) \)

(why is there no covariance term?)

\[
\text{Var}(Y) = \text{Var}(\hat{Y}) + \text{Var}(e)
\]

- This is a neat result. It says that the variance of Y is due to two components, the part due to variation in the X and a part that is unrelated to X, the error e.
- If the X’s predicted perfectly then all the variance in Y would be explained by the X’s and \( \text{Var}(Y) = \text{Var}(\hat{Y}) \).
- If the X’s don’t predict Y at all then \( \text{Var}(\hat{Y}) = 0 \) and \( \text{Var}(Y) = \text{Var}(e) \).
The $R^2$, measures goodness of fit:

$$R^2 = \frac{\text{Var}(\hat{Y})}{\text{Var}(Y)}$$

proportion of variation in $Y$ explained by the X's

$R^2$ is between 0 and 1.

The closer $R^2$ is to 1, the better the fit.

---

Back to the regression output

Here is the $R^2$

Results of simple regression for Price

Summary measures

| Multiple R | 0.9092 |
| R-Square  | 0.8267 |
| SE of Est  | 14.1384 |

ANOVA table

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>p-value</th>
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Regression coefficients

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<th>Std Err</th>
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<tr>
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</table>
8. The Simple Linear Regression Model

8.1 The “True” Model
8.2 Sampling Distributions of the regression model and confidence intervals
8.3 Hypothesis Testing

8.1 The true model

• We now describe the “true” relationship between X and Y or the “true regression model”.
• In the last couple of sections of the notes, we considered the example of estimating the Normal(μ, σ²) model with parameters μ and σ².
• We viewed \( \bar{x} \) and \( s_x^2 \) as estimates of μ and σ².
• We now view \( b_0 \) and \( b_1 \) as estimates of parameters of a “true model”.
• So what is the true regression model?
Our model is:

\[ Y = \beta_0 + \beta_1 X + \varepsilon \]

The true line describing the linear pattern

How far off the line the points tend to be (everything about Y not captured by X)

\( \beta_0 \) is the true intercept

\( \beta_1 \) is the true slope

We assume that \( \varepsilon \) has a normal:

\[ \varepsilon \sim N(0, \sigma^2) \]

Saying that \( \varepsilon \) is a normally distributed random variable is a statement about the possible values that \( \varepsilon \) could turn out to be.

mean of \( \varepsilon \) is zero

(variance of \( \varepsilon \) is \( \sigma^2 \))

(sometimes Y is above the line, sometimes Y is below the line, on average Y is on the line)

and \( \varepsilon \) is independent of \( X \).

If \( \sigma^2 \) is small, \( \varepsilon \) tends to be small (close to zero). If \( \sigma^2 \) is large, \( \varepsilon \) tends to be large (far from zero).
A picture of the model: \( Y = \beta_0 + \beta_1 X + \epsilon \)

Given \( X=x_1 \)

You can see the role played by \( \sigma \)

If \( \sigma \) is small...

If \( \sigma \) is large...
The model has three unknown parameters:
\[
Y = \beta_0 + \beta_1 X + \varepsilon
\]
\[
\varepsilon \sim N(0, \sigma^2)
\]

We estimate these parameters from the data.

In fact, this is what we have ...

From the data we have to estimate the true line \((\beta_0 \text{ and } \beta_1)\) and \(\sigma\).
8.2 Sampling Distribution for the regression model and confidence intervals

• It turns out that both \( b_0 \) and \( b_1 \) are unbiased estimates of \( \beta_0 \) and \( \beta_1 \) so that \( E(b_0) = \beta_0 \) and \( E(b_1) = \beta_1 \).

• The next question is how wrong might they be?

• We need to know the standard deviation (or standard error) of \( b_0 \) and \( b_1 \) denoted by \( s_{b_0} \) and \( s_{b_1} \).

Back to the housing regression:

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<td>25.6771</td>
</tr>
</tbody>
</table>
It turns out that the standardized distance that \( b_j \) lies away from the true parameter \( \beta_j \) follows a the \( t_{n-2} \) distribution (“t distribution with n-2 degrees of freedom”):

\[
\frac{b_j - \beta_j}{s_{b_j}} \sim t_{n-2}
\]

**The confidence intervals and hypothesis tests are based on this key result.**

Thus, a 95% confidence interval for \( \beta_j \) is

\[
\left( b_j - t_{n-2,.025} s_{b_j}, b_j + t_{n-2,.025} s_{b_j} \right)
\]

- This is the same notation as before, specifically, we would use
  
  \( = \text{tinv}(.05,n-2) \) to obtain \( t_{n-2,.025} \)
Back to the housing regression:

<table>
<thead>
<tr>
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<td>7.8739</td>
<td>0.0000</td>
<td>25.6771</td>
<td>45.0948</td>
</tr>
</tbody>
</table>

Goal: Form a 95% confidence interval for $\beta_1$.

\[
\hat{\beta}_1 = 35.4 \\
\hat{\sigma}_b = 4.5 \\
t_{0.025} = t_{0.025} = 2.16
\]

Calculation:

\[
35.4 \pm (2.16)(4.5) = 35.4 \pm 9.7
\]

95% confidence interval for $\beta_1$: $(25.7, 45.1)$

8.3 Hypothesis Testing

Suppose you are interested in testing whether the true slope parameter, $\beta_1$, is equal to a particular value.

Ex: Is the “beta coefficient” from the market model equal to 1?

Maybe you want to test whether X affects Y in any simple linear regression model, we would test whether $\beta_1$ is equal to zero.
Example

In finance, a well known model for pricing assets regresses stock returns against returns of some market index, such as the S&P 500. The slope of the regression line is referred to as the asset’s “beta”. In your finance course you will learn that asset’s with high betas tend to be viewed as more risky than assets with low betas.

Facts:
The market portfolio (S&P500) has a beta of 1
A portfolio with a beta larger than 1 should have a higher average return than the market (S&P500)
A portfolio with a beta less than 1 should have a smaller average return than the market (S&P500).

Form the following null hypothesis:

\[ H_0 : \beta = \beta_0 \]

The alternative hypothesis is defined as the opposite of the null hypothesis:

\[ H_a : \beta \neq \beta_0 \]

For the market model example, the null and alternative hypotheses are:

\[ H_0 : \beta = 1 \text{ and } H_a : \beta \neq 1 \]
To test $H_0 : \beta_i = \beta_i^0$, form the **t-statistic**:

$$
t = \frac{b_i - \beta_i^0}{s_{b_i}}
$$

The intuition is exactly as before.

The top is just the difference between the estimate and the claimed value.

The bottom is the standard error that accounts for the accuracy of our estimate.

If the null hypothesis is true, the t-statistic should be small (in absolute value).

If the null hypothesis is false, the t-statistic should be large (in absolute value).

### The t-test

Reject $H_0$ at the 5% level if

$$
|t| = \left| \frac{b_i - \beta_i^0}{s_{b_i}} \right| > t_{n-2,.025}
$$
Beta estimate for IBM (n = 240):

<table>
<thead>
<tr>
<th>Regression coefficients</th>
<th>Coefficient</th>
<th>Std Err</th>
<th>t-value</th>
<th>p-value</th>
<th>Lower limit</th>
<th>Upper limit</th>
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<td>8.7891</td>
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<td>0.6896</td>
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</tr>
</tbody>
</table>

Test #1: *Test the null hypothesis that there is no relationship between the return on the IBM & the market return.*

\[ H_0: \beta_1 = 0 \text{ and } H_a: \beta_1 \neq 0 \]

\[ t = \frac{b_1 - 0}{s_{b_1}} = \frac{0.889}{0.101} = 8.78 \]

Large sample so t-cutoff is the same as Normal(0,1), or 2!

Reject \( H_0 \) at the 5% level.
Test #2: Test the null hypothesis that the IBM has the same risk as the market portfolio (i.e. test that the beta is 1).

\[ H_0: \beta_1 = 1 \quad \text{and} \quad H_a: \beta_1 \neq 1 \]

t-statistic:

\[
t = \frac{\hat{\beta}_1 - 1}{s_{\hat{\beta}}} = \frac{-1.111}{0.1011} = -1.0989
\]

Fail to reject \( H_0 \) at the 5% level.

P-values are calculated just as before.

\[
p\text{-value} = \Pr\left( | \hat{t}_{n-2} | \geq | t | \right)
\]

- \( \hat{t}_{n-2} \) random variable from \( t_{n-2} \) distribution
- \( t \) calculated t-statistic
The p-value for testing $\beta_1 = 0$ is part of Excel’s regression output (under the column labeled “p”).

Compute the p-value for testing $\beta_1 = 1$ from the IBM regression:

\[ pvalue = \Pr(\left| t_{238} \right| \geq 1.0989) \]

\[ = \text{tdist}(1.0989,238,2) = .273 \]

So, the p-value is 27.3%.

---

**RECALL**

Loosely speaking:

**Small p-value means we reject!**

More precisely:

**If the p-value < .05, then we reject at level .05!**

In general:

**If the p-value < $\alpha$, then we reject at level $\alpha$!**
Adding more X’s in the prediction problem, the multiple regression model

• Clearly there is more information that is useful in predicting the sales price of a house than just the size.
• We need to expand our model to allow for this possibility.
• This will be called the multiple regression model.

The multiple regression model

9.1 The multiple regression model.
9.3 The distribution of the multiple regression parameters.
9.1 Multiple regression model

• So far we have considered modeling and prediction when we have one $Y$ variable and one $X$.
• Often we might think that there is more than one variable that would be useful in predicting (the size of the house is surely not the only variable that carries information about the expected sales price!)
• More information should lead to **more precise predication**.

The “true” model with $k$ independent ($X$) variables:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k + \varepsilon$$

$$\varepsilon \sim N(0, \sigma^2) \text{ i.i.d.}$$

The conditional mean,

$$E(Y|X_1, X_2, \ldots, X_k) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k,$$

is the part of $Y$ related to the $X$'s.

The residual, $\varepsilon$, is independent of **all** the $X$ variables!
Once there is more than one X variable, it’s difficult to plot the data and/or the regression “line.”

One could use a 3-D plot for the case of two X variables, but anything more than two variables is basically hopeless...

Interpreting the coefficients:

\[ \beta_j = \frac{\Delta E(Y \mid X_1, \ldots, X_k)}{\Delta X_j} \]

the average (or expected) change in Y for a one unit change in \( X_j \), holding all of the other independent variables fixed

R-squared

- Rearranging the terms in the definition of the residual: \( Y_i = \hat{Y}_i + e_i \) and \( Var(Y) = Var(\hat{Y}) + Var(e) \).

- The R-squared has the same interpretation as before and is defined as:

\[ R^2 = \frac{Var(\hat{Y})}{Var(Y)} \]

- It answers the question of “what fraction of the variation in the Y’s is explained by the X’s”.

[Diagram of regression line with coefficients explained]
Given $b_0, b_1, \ldots, b_k$, the estimate of the variance of the true residual ($\sigma^2$) is

$$s^2 = \frac{1}{n - k - 1} \sum_{i=1}^{n} e_i^2$$

and the estimate of the standard deviation ($\sigma$) is

$$s = \sqrt{\frac{1}{n - k - 1} \sum_{i=1}^{n} e_i^2}$$

Fact: $E(s^2) = \sigma^2$ (unbiased)

---

Regression output of son’s height on mom’s height and dad’s height

<table>
<thead>
<tr>
<th>Results of multiple regression for SHGT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Summary measures</td>
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<tr>
<td>Multiple R 0.5083</td>
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<td>Explained</td>
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</tr>
<tr>
<td>$b_1$</td>
</tr>
<tr>
<td>$b_2$</td>
</tr>
</tbody>
</table>
9.2 Distribution of $b_j$ in the multiple regression

Fact: $\text{E}(b_0) = \beta_0$, $\text{E}(b_1) = \beta_1$, …, $\text{E}(b_k) = \beta_k$

Fact: $\frac{b_j - \beta_j}{s_{b_j}} \sim t_{n-k-1}$

accounts for the fact that (k+1) parameters are being estimated

This fact can be used to construct confidence intervals and perform t-tests.

(1) Confidence Intervals

95% confidence interval is:

$$(b_j - t_{n-k-1,0.025} s_{b_j}, b_j + t_{n-k-1,0.025} s_{b_j})$$
t-Tests and p-values

Test \( H_0 : \beta_j = \beta_j^0 \) vs. \( H_a : \beta_j \neq \beta_j^0 \)

Reject \( H_0 \) at the 5% level if \( t = \frac{\hat{\beta_j} - \beta_j^0}{s_{\beta_j}} \geq t_{n-k-1, 0.025} \)

\[
p-value = \Pr \left( t_{n-k-1} \geq t \right)
\]

random variable from \( t_{n-k-1} \) distribution

calculated t-statistic

Example: Regression of male MBA student height on parents’ heights (n = 76)

<table>
<thead>
<tr>
<th>Regression coefficients</th>
<th>Coefficient</th>
<th>Std Err</th>
<th>t-value</th>
<th>p-value</th>
<th>Lower limit</th>
<th>Upper limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>29.0036</td>
<td>8.4198</td>
<td>3.4447</td>
<td>0.0010</td>
<td>12.2230</td>
<td>45.7842</td>
</tr>
<tr>
<td>MHGT</td>
<td>0.3601</td>
<td>0.1092</td>
<td>3.2966</td>
<td>0.0015</td>
<td>0.1424</td>
<td>0.5778</td>
</tr>
<tr>
<td>FHGT</td>
<td>0.2726</td>
<td>0.1063</td>
<td>2.5641</td>
<td>0.0124</td>
<td>0.0607</td>
<td>0.4846</td>
</tr>
</tbody>
</table>

"partial" effects are significantly different from zero

95% confidence interval for MomHgt estimate:

\((0.36 - t_{73,0.025} (0.109), 0.36 + t_{73,0.025} (0.109)) \approx (0.14, 0.58)\)

95% confidence interval for DadHgt estimate:

\((0.273 - t_{73,0.025} (0.106), 0.273 + t_{73,0.025} (0.106)) \approx (0.061, 0.485)\)