Time Varying Volatility

- **CONDITIONAL VOLATILITY** is the standard deviation of the unpredictable part of the series.
- We define the conditional variance as:

\[ \sigma_t^2 = E \left( \left( y_t - E \left( y_t \mid F_{t-1} \right) \right)^2 \mid F_{t-1} \right) = E \left( \epsilon_t^2 \mid F_{t-1} \right) \]

- If the mean is time varying then we want to subtract off the conditional mean at time \( t \) when we calculate the variance.
• We will focus on the volatility now. If there is a time varying mean, we will consider the series to be demeaned and we will model the variance of $\varepsilon_t$ where $\varepsilon_t = r_t - \mu_t$

• Notice that if the mean $\mu_t$ is zero then we have $r_t = \varepsilon_t$

• For now, we will set $\mu_t$ to zero, later, we will consider joint estimation of a conditional mean and variance equation.

Consider equity returns

• Let’s first take a look at historical volatility. Then we will consider models that capture the time varying features observed in the data.
Volatility in Equities and indexes: S&P500
CHARACTERISTICS OF FINANCIAL RETURNS

- ALMOST UNPREDICTABLE
  - EFFICIENT MARKET HYPOTHESIS
- SURPRISINGLY LARGE NUMBER OF EXTREMES
  - FAT TAIL DISTRIBUTIONS
- PERIODS OF HIGH AND LOW VOLATILITIES
  - VOLATILITY CLUSTERING
- WHY DOES VOLATILITY DO THIS?
- WHAT CHANGES ASSET PRICES?
CHECK IT OUT

• HOW TO CHECK FOR EXCESSIVE EXTREMES

• HOW TO CHECK FOR VOLATILITY CLUSTERING?

HISTORICAL VOLATILITY

• Estimate the standard deviation of a random variable

\[ \hat{\sigma} = \sqrt{\frac{252}{T-K} \sum_{j=T-K}^{T} r_j^2 / K} \]

• What assumptions do we need?
  – Choose K small so that the variance is constant
  – Choose K large to make the estimate as accurate as possible
• Funny boxcars and shadow volatility movements!!
EXPONENTIAL SMOOTHING

- Volatility Estimator used by RISKMETRICS
- \[ \sigma^2_t = \lambda \sigma^2_{t-1} + (1 - \lambda) r^2_{t-1} \]
- Updating
- AN EXAMPLE
- WEAKNESSES
  - How to choose lambda
  - No mean reversion

II ARCH/GARCH MODELS

- GARCH VOLATILITY
- FORECASTING WITH GARCH
- ESTIMATING AND TESTING GARCH
- MANY MODELS
The ARCH Model

• The ARCH model of Engle (1982) is a family of specifications for the conditional variance.
• The $q^{th}$ order ARCH or ARCH(q) model is

$$h_t = \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}^2$$

• Where in the GARCH notation $h_t$ is the conditional variance $\sigma_t^2 = E(r_t^2 | F_{t-1})$.

Extensions

• GENERALIZED ARCH (Bollerslev) a most important extension
• Tomorrow’s variance is predicted to be a weighted average of the
  — Long run average variance
  — Today’s variance forecast
  — The news (today’s squared return)
GARCH

\[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]

- Generalization of Exponential Smoothing
- Generalization of ARCH
- Generalization of constant volatility

UPDATING

- Suppose the model is:
  
  \[ h_t = .00001 + .05 r_{t-1}^2 + .9 h_{t-1} \]

- And today annualized volatility is 20% and the market return is -3%, what is my estimate of tomorrow’s volatility from this model?
REPEAT STARTING AT T=1

• IF WE KNOW THE PARAMETERS
• AND SOME STARTING VALUE FOR \( h_1 \), WE CAN CALCULATE THE ENTIRE HISTORY OF VOLATILITY FORECASTS
• OFTEN WE USE A SAMPLE VARIANCE FOR \( h_1 \).

GARCH(p,q)

• The Generalized ARCH model of Bollerslev(1986) is an ARMA version of this model. GARCH(p,q) is

\[
h_t = \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}
\]
Asymmetric Volatility

• Often negative shocks have a bigger effect on volatility than positive shocks
• Nelson(1987) introduced the EGARCH model to incorporate this effect.
• I will use a Threshold GARCH or TARCH

\[ h_t = \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}^2 + \sum_{j=1}^{q} \gamma_j r_{t-j}^2 I_{(r_{t-j} < 0)} + \sum_{j=1}^{p} \beta_j h_{t-j}, \]

NEW ARCH MODELS

• GJR-GARCH
• TARCH
• STARCH
• AARCH
• NARCH
• MARCH
• SWARCH
• SNPARCH
• APARCH
• TAYLOR-SCHWERT
• FIGARCH
• FIEGARCH
• Component
• Asymmetric Component
• SQGARCH
• CESGARCH
• Student t
• GED
• SPARCH
• Autoregressive Conditional Density
• Autoregressive Conditional Skewness
CONFIDENCE INTERVALS

UNCONDITIONAL, OR LONG RUN, OR AVERAGE VARIANCE

- WHAT IS $E(r^2)$?

\[
\sigma^2 \equiv E\left( r_t^2 \right) = E\left( E\left( r_t^2 \mid past \right) \right) \equiv E(h_t)
\]

by the Law of Iterated Expectations

- $E(h_t) = \omega + \alpha E(r_{t-1}^2) + \beta E(h_{t-1})$

- $\sigma^2 = \omega + (\alpha + \beta) E(h) = \omega + (\alpha + \beta) \sigma^2$

- $\sigma^2 = \frac{\omega}{1 - \alpha - \beta}$
The GARCH Model Again

- \[ r_t = \varepsilon_t \]
- \[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]
  \[ = \sigma^2 (1 - \alpha - \beta) + \alpha r_{t-1}^2 + \beta h_{t-1} \]

- The variance of \( r_t \) is a weighted average of three components
  - a constant or unconditional variance
  - yesterday’s forecast
  - yesterday’s news

Multi-step forecasts

- One-step: \[ E_t(r_{i,t}) = h_{i,t} = \omega + \alpha \varepsilon_t^2 + \beta h_t \]
  or \[ h_{i,t} - \sigma^2 = \alpha (r_{i,t}^2 - \sigma^2) + \beta (h_t - \sigma^2) \]

Iterating one step forward we get:

\[ h_{i,t+1} - \sigma^2 = \alpha (r_{i,t}^2 - \sigma^2) + \beta (h_{i,t} - \sigma^2) \]

Now take expectations with respect to time \( t \):

\[ E_t(h_{i,t+2} - \sigma^2) = \alpha E_t(r_{i,t}^2 - \sigma^2) + \beta (h_{i,t} - \sigma^2) = (\alpha + \beta)(h_{i,t} - \sigma^2) \]
• So two-step is: 
\[ E_t(h_{t+2} - \sigma^2) = (\alpha + \beta)(h_{t+1} - \sigma^2) \]

• Iterating again and taking expectations with respect to time t:
\[ E_t(h_{t+3} - \sigma^2) = (\alpha + \beta)(h_{t+2} - \sigma^2) \]
\[ E_t[E_t(h_{t+3} - \sigma^2)] = E_t(h_{t+3} - \sigma^2) = (\alpha + \beta)E_t(h_{t+2} - \sigma^2) = (\alpha + \beta)^2(h_{t+1} - \sigma^2) \]

• More generally: 
\[ E_t[(h_{t+k} - \sigma^2)] = (\alpha + \beta)^{k-1}(h_{t+1} - \sigma^2) \]

or 
\[ E_t(h_{t+k}) = \sigma^2 + (\alpha + \beta)^{k-1}(h_{t+1} - \sigma^2) \]

---

**MEAN REVERTING VOLATILITY**

• Forecasts converge to the same value no matter what the current volatility
\[ E(h_t^k) = \sigma^2 + (\alpha + \beta)^{k-1}\left(h_{t+1} - \sigma^2\right) \]
\[ E(h_t^k) \rightarrow \sigma^2 \text{ if } \alpha + \beta < 1 \]

• LITTLE UPDATING FOR LONG HORIZON VOLATILITY
Monotonic Term Structure of Volatility

\[ \sigma \]

FORECAST PERIOD

FORECASTING WITH GARCH-
another derivation

\[
r_t^2 = \omega + (\alpha + \beta)r_{t-1}^2 - \beta(r_{t-1}^2 - h_{t-1}) + (r_t^2 - h_t)
\]

- GARCH(1,1) can be written as ARMA(1,1)
- The autoregressive coefficient is \((\alpha + \beta)\)
- The moving average coefficient is \(-\beta\)
• In general, a GARCH(p,q) model can be expressed as an ARMA(max(p,q),p) model for the squared returns.

FORECASTING VOLATILITY
DOW JONES SINCE 1990

Dependent Variable: DJRET
Method: ML - ARCH (Marquardt) - Normal distribution

Date: 01/10/08  Time: 13:42
Sample: 1/02/1990 1/04/2008
Included observations: 4541
Convergence achieved after 15 iterations

GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.000027</td>
<td>0.000119</td>
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Variance Equation

<table>
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<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1.00E-06</td>
<td>1.37E-07</td>
<td>7.290125</td>
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<tr>
<td>RESID(-1)^2</td>
<td>0.064459</td>
<td>0.004082</td>
<td>15.79053</td>
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<tr>
<td>GARCH(-1)</td>
<td>0.925645</td>
<td>0.005025</td>
<td>184.2160</td>
</tr>
</tbody>
</table>

R-squared    -0.000371  Mean dependent var 0.000338
Adjusted R-squared -0.001032  S.D. dependent var 0.009795
S.E. of regression 0.009800  Akaike info criterion -6.640830
Sum squared resid 0.435759  Schwarz criterion -6.635174
Log likelihood 15082.00  Hannan-Quinn criter. -6.638838
Durbin-Watson stat 2.001439

VOLGARCH

![VOLGARCH Chart](chart.png)
FORECAST

VOLFORECAST

DJSD  DJSD1  DJSD2  DJSD3  DJSD4  DJSD5
EXOGENOUS VARIABLES IN A GARCH MODEL

• Include predetermined variables into the variance equation
• Easy to estimate and forecast one step
• Multi-step forecasting is difficult
• Timing may not be right

\[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} + \gamma z_{t-1} \]

EXAMPLES

• Non-linear effects
• Deterministic Effects
• News from other markets
  – Heat waves vs. Meteor Showers
  – Other assets
  – Implied Volatilities
  – Index volatility
• MacroVariables or Events
PARAMETER ESTIMATION

• MLE on white board
• Joint ARCH/mean estimation.

PLAUSIBLE ANSWERS

• WE EXPECT ALL THREE PARAMETERS OF A GARCH(1,1) TO BE POSITIVE.
• WE EXPECT THE SUM OF ALPHA AND BETA TO BE VERY CLOSE TO ONE BUT LESS THAN ONE.
• WE EXPECT THE UNCONDITIONAL VARIANCE TO BE CLOSE TO THE DATA VARIANCE.
DID THE ESTIMATION ALGORITHM CONVERGE?

• Generally the software will reliably find the maximum of the likelihood function and will report it.
• Sometimes it does not. You may get silly values. What then?
  – Check with other starting values
  – Check with other iterations
  – Scale the data so the numbers are not so small
• Often the problem is the data. Look for outliers or peculiar features.
  – Use longer data set

NORMALLITY

• THIS ESTIMATION METHOD IS OPTIMAL IF THE ERRORS ARE NORMAL AND IF THE SAMPLE IS LARGE AND THE MODEL IS CORRECT.
• IT IS STILL GOOD WITHOUT NORMALITY
• BUT OTHER ESTIMATORS COULD BE BETTER SUCH AS STUDENT-T.
ERRORS

• THE ERRORS MUST HAVE VARIANCE 1
• THEY COULD BE NORMAL
• THEY MIGHT HAVE FATTER TAILS LIKE THE STUDENT –T OR GENERALIZED EXPONENTIAL
• IN GENERAL WE CAN THINK OF THE GARCH MODEL AS: \( r_t = \sqrt{h_t} z_t \), where \( z_t \) is iid with \( \text{var}(z_t) = 1 \).

STUDENT-T ERRORS

• Assume that: \( z_t \sim \text{Student-t}, \nu \)
• Because \( V(z) = \nu / (\nu - 2) \), \( V(\frac{z}{\sqrt{\nu / (\nu - 2)}}) = 1 \)
• Then let \( r_t = \sqrt{h_t} \frac{z_t}{\sqrt{\nu / (\nu - 2)}} \), or \( \frac{\sqrt{\nu / (\nu - 2)}}{\sqrt{h_t}} r_t = z_t \),
• Perform MLE with standardized t-distribution
COMPARE MODELS

• Models which achieve the highest value of the log likelihood are preferred.
• If they have different numbers of parameters – this is not a fair comparison.
  — Use AIC or BIC (Schwarz) instead. The smallest value is best.

DIAGNOSTIC CHECKING

• Time varying volatility is revealed by volatility clusters
• These are measured by the Ljung Box statistic on squared returns
• The standardized returns $z_t = r_t / \sqrt{h_t}$ no longer should show significant volatility clustering
WHAT IS THE BEST MODEL?

• The most reliable and robust is GARCH(1,1)
• A student-t error assumption gives better estimates of tails.
• For equities asymmetry is almost always important.
• For long term forecasts, a component model is often needed.
• Even better is a model which incorporates economic variables

III NON-NORMAL ERRORS and GARCH:

• VALUE AT RISK
• GARCH
• ASYMMETRIC VOLATILITY
• DOWNSIDE RISK
• BUBBLES AND CRASHES
Value at Risk

- For a portfolio the future value is uncertain
- VaR is a number of $ that you can be 99% sure, is worse than what will happen.
- It is the 99% of the loss distribution (or the 1% quantile of the gain distribution)
- Simple idea, but how to calculate this?

PREDICTIVE DISTRIBUTION OF PORTFOLIO GAINS

1%  $ GAINS ON PORTFOLIO
HISTORICAL VaR

- If History repeats, look at worst outcomes in the past
- For example, Dow Jones over the last year.
- On a $1,000,000 portfolio, the 99% VaR is ?

HISTOGRAM OF D.J. GAINS

1% quantile = -0.0154
HISTORICAL D.J. VaR

- If I use 2 years of data, it is $20,339
- With 3 years, it is $29,087
- And with 75 years it is $33,748
- Which is more accurate?

VOLATILITY BASED VaR

- With a good volatility forecast, predict the standard deviation of tomorrows return.
- Assume a Normal Distribution. Then

\[ \text{VaR is } 2.33 \times \sigma_t \]

- But what do we use for the volatility?
- GARCH forecasts!
- Other volatility estimates?
GARCH MODEL FOR DJ

- USE FOR EXAMPLE DATA FOR 10 YEARS (95-05)
- FORECAST OUT OF SAMPLE AND RECORD THE DAILY STANDARD DEVIATION
- MULTIPLY BY 2.33
- WE GET

RESULTS

<table>
<thead>
<tr>
<th>DATE</th>
<th>RETURN</th>
<th>DAILY SD</th>
<th>VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005-01-07</td>
<td>-0.001783</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>2005-01-10</td>
<td>NA</td>
<td>0.006166</td>
<td>14367.77</td>
</tr>
</tbody>
</table>
VOLATILITY BASED VaR WITH STUDENT-T ERRORS

• Assume that:
  \[ z_i \sim Student - t, v \]

• Because
  \[ V(z) = v/(v-2), V\left(\frac{z}{\sqrt{v/(v-2)}}\right) = 1 \]

• Then let
  \[ r_i = \sqrt{h_i} \cdot \frac{z_i}{\sqrt{v/(v-2)}} \]

• And estimate volatility and the shape of the error distribution jointly.

• In EViews =@qtdist(.01,v)/sqr[v/(v-2)]

STUDENT-T RESULTS

<table>
<thead>
<tr>
<th>DATE</th>
<th>RETURN</th>
<th>DAILY SD</th>
<th>VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005-01-07</td>
<td>-0.001783</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>2005-01-10</td>
<td>NA</td>
<td>0.006260</td>
<td>155874</td>
</tr>
</tbody>
</table>

• GARCH WITH STUDENT T ERRORS
  - C 1.01E-06 3.20E-07 3.147330 0.0016
  - RESID(-1)^2 0.063884 0.009483 6.736429 0.0000
  - GARCH(-1) 0.929008 0.009920 93.65090 0.0000
  - T-DIST. DOF 8.839721 1.240570 7.125529 0.0000
  • .01 QUANTILE OF UNIT STUDENT –T DISTRIBUTION(8.8DF) IS –2.49
VOLATILITY BASED VaR WITHOUT NORMALITY or T

• What is the right multiplier for the true distribution? Maybe neither the normal nor the student t are correct!

• If: \( r_t = \sqrt{h_t} z_t, \quad z_t \sim i.i.d. \)

• Then 1% quantile of the standardized residuals should be used. This is the bootstrap estimator or Hull and White’s volatility adjustment.

HISTOGRAM OF STANDARDIZED RESIDUALS

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Value</th>
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<tbody>
<tr>
<td>Mean</td>
<td>-0.040388</td>
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<tr>
<td>Median</td>
<td>-0.021369</td>
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<tr>
<td>Maximum</td>
<td>3.036714</td>
</tr>
<tr>
<td>Minimum</td>
<td>-6.058763</td>
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<tr>
<td>Std. Dev.</td>
<td>0.999691</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.383229</td>
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<tr>
<td>Kurtosis</td>
<td>4.474130</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>289.8543</td>
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<tr>
<td>Probability</td>
<td>0.000000</td>
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</tbody>
</table>

Series: GARCHRESID
Sample 1/09/1995 1/20/2005
Observations 2520
BOOTSTRAP VaR

<table>
<thead>
<tr>
<th>DATE</th>
<th>RETURN</th>
<th>DAILY SD</th>
<th>VaR</th>
</tr>
</thead>
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<tr>
<td>2005-01-07</td>
<td>-0.001783</td>
<td>NA</td>
<td>NA</td>
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<tr>
<td>2005-01-10</td>
<td>NA</td>
<td>0.006166</td>
<td>15724.38</td>
</tr>
</tbody>
</table>

OVERVIEW AND REVIEW

- HISTORICAL QUANTILES – RESULT IS SENSITIVE TO SAMPLE INCLUDED
- VOLATILITY BASED
  - RESULT IS SENSITIVE TO THE ERROR DISTRIBUTION
  - NORMAL UNDERSTATES EXTREME RISK
  - T AND BOOTSTRAP ARE BETTER.
  - RESULTS ARE NOT SENSITIVE TO THE SAMPLE INCLUDED
Asymmetric Volatility Models and the distribution of returns.

Time varying volatility induces excess kurtosis in the unconditional distribution of returns.

• Let \( z_t \) be iid with mean zero and variance 1

\[
E(r_t^4) = E \left( \left( \sqrt{h_t} z_t \right)^4 \right) = E(h_t^2) E(z_t^4) > \sigma^4 \frac{E(z_t^4)}{1^4} = \sigma^4 \kappa_z
\]

\[
\kappa_r = \frac{E(r_t^4)}{\sigma^4} > \kappa_z
\]

Where \( \kappa_z \) is the kurtosis of \( z \) and \( \kappa_r \) is the kurtosis of the returns.

• Hence the kurtosis of returns is greater than the conditional kurtosis, the kurtosis of \( z \).
• Bollerslev (1985) shows that if the z’s are Normal, then the excess kurtosis for the returns of a GARCH(1,1) is given by:

\[
\kappa^g = \frac{6\alpha_1^2}{1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2}
\]

• Furthermore, Bai, Russell, and Tiao (2003) show that if z’s are non-normal then the excess kurtosis is given by:

\[
\kappa_r = \frac{\kappa_z + \kappa^g + \frac{5}{6}\kappa_z\kappa^g}{1 - \frac{1}{6}\kappa_z\kappa^g}
\]

• where \(\kappa^g\) is the implied excess kurtosis when the returns are normal (as in the previous slide). \(\kappa_z\) is the excess kurtosis of z.
ASYMMETRIC VOLATILITY

- Positive and negative returns might have different weights.
- For example:

\[ h_t = \omega + \alpha_1 r_{t-1}^2 I_{r_{t-1} > 0} + \alpha_2 r_{t-1}^2 I_{r_{t-1} < 0} + \beta h_{t-1} \]

\[ h_t = \omega + \alpha r_{t-1}^2 + \gamma r_{t-1}^2 I_{r_{t-1} < 0} + \beta h_{t-1} \]

- We typically find for equities that

\[ \alpha_2 > \alpha_1 \] or equivalently \[ \gamma > 0 \]

NEWS IMPACT CURVE

TOMORROWS VARIANCE

TODAY'S NEWS = RETURNS
Other Asymmetric Models

**EGARCH: NELSON(1989)**

$$\log(h_t) = \omega + \beta \log(h_{t-1}) + \alpha \frac{r_{t-1}}{\sqrt{h_{t-1}}} + \gamma \frac{r_{t-1}}{\sqrt{h_{t-1}}}$$

**NGARCH: ENGLE(1990)**

$$h_t = \omega + \alpha (r_{t-1} - \gamma)^2 + \beta h_{t-1}$$

**PARTIALLY NON-PARAMETRIC**

**ENGLE AND NG(1993)**

![Volatility vs. News Graph](image)
WHERE DOES ASYMMETRIC VOLATILITY COME FROM?

• LEVERAGE - As equity prices fall the leverage of a firm increases so that the next shock has higher volatility on stock prices.
  – This effect is usually too small to explain what we see.

• RISK AVersion– News of a future volatility event will lead to stock sales and price declines. Subsequently, the volatility event will occur. Since events are clustered, any news event will predict higher volatility in the future.
  – This effect is more plausible on broad market indices since these have systematic risk.

BACK TO VALUE AT RISK

• FIND QUANTILE OF FUTURE RETURNS
  – One day in advance
  – Many days in advance
• REGULATORY STANDARD IS 10 DAY 1% VaR.
MULTI-DAY RETURN DISTRIBUTION AND VaR

• What is the risk over 10 days if you do no more trading? Clearly this is greater than for one day.

• Now we need the distribution of multi-day returns.

10 Day VaR

• If volatility were constant, then the multi-day volatility would simply require multiplying by the square root of the days.
• With normality and constant variance this becomes 7.36 or sqrt(10)*2.33
• VaR is 7.36 * sigma
  – What is sigma?
MULTI-DAY HORIZONS

• Because volatility is dynamic and asymmetric, the lower tail is more extreme and the VaR should be greater.

TWO PERIOD RETURNS

• Two period return is the sum of two one period continuously compounded returns
• Look at binomial tree version
• Asymmetry gives negative skewness
MULTIPLIER FOR 10 DAYS

• For a 10 day 99% value at risk, conventional practice multiplies the daily standard deviation by 7.36
• For the same multiplier with asymmetric GARCH it is simulated from the example to be 7.88
• Bootstrapping from the residuals the multiplier becomes 8.52

CALCULATION BY SIMULATION

• EVALUATE ANY MEASURE BY REPEATEDLY SIMULATING FROM THE ONE PERIOD CONDITIONAL DISTRIBUTION: $f_t(r_{t+1})$

• METHOD:
  – Draw $r_{t+1}$
  – Update density and draw observation $t+2$
  – Continue until T returns are computed.
  – Repeat many times
  – Compute measure of downside risk
ESTIMATE TARCH MODEL

<table>
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<tr>
<th>VARIABLE</th>
<th>COEF</th>
<th>STERR</th>
<th>T-STAT</th>
<th>P-VALUE</th>
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<td>GARCH(-1)</td>
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<table>
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<td>0.006835</td>
</tr>
<tr>
<td>2005-01-10</td>
<td>0.006726</td>
</tr>
</tbody>
</table>

**TARCH STANDARD DEVIATIONS**

![Graph showing TARCH standard deviations over time](chart.png)
TARCH STANDARD DEVIATIONS

DOWNSIDE RISK
DOWNSIDE RISK

• With Asymmetric Volatility, the multi-period returns are asymmetric with a longer left tail.
• For long horizons, the central limit theorem will reduce this effect and returns will be approximately normal.
• This is observed in data too.

1 DAY RETURNS ON D.J.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Series</td>
<td>DJRET</td>
</tr>
<tr>
<td>Sample</td>
<td>1/03/1995 1/20/2005</td>
</tr>
<tr>
<td>Observations</td>
<td>2524</td>
</tr>
<tr>
<td>Mean</td>
<td>0.000403</td>
</tr>
<tr>
<td>Median</td>
<td>0.000557</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.061547</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.074549</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.011179</td>
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<tr>
<td>Skewness</td>
<td>-0.250246</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>7.024450</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>1729.644</td>
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<tr>
<td>Probability</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
10 DAY RETURNS ON D.J.

Series: @MOVSUM(DJRET,10)
Sample 1/03/1995 1/20/2005
Observations 2524
Mean 0.004091
Median 0.006289
Maximum 0.153400
Minimum -0.189056
Std. Dev. 0.033474
Skewness -0.619562
Kurtosis 6.014598
Jarque-Bera 1117.209
Probability 0.000000

SKEWNESS OF MULTIPERIOD RETURNS

-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1
25 50 75 100 125 150 175 200 225

- SKEW_ALL
- SKEW_TRIM
- SKEW_PRE
- SKEW_POST
EVIDENCE FROM DERIVATIVES

• THE HIGH PRICE OF OUT-OF-THE-MONEY EQUITY PUT OPTIONS IS WELL DOCUMENTED
• THIS IMPLIES SKEWNESS IN THE RISK NEUTRAL DISTRIBUTION
• MUCH OF THIS IS PROBABLY DUE TO SKEWNESS IN THE EMPIRICAL DISTRIBUTION OF RETURNS.
• DATA MATCHES EVIDENCE THAT THE OPTION SKEW IS ONLY POST 1987.

MATCHING THE STYLIZED FACTS

• ESTIMATE DAILY MODEL
• SIMULATE 250 CUMULATIVE RETURNS 10,000 TIMES WITH SEVERAL DATA GENERATING PROCESSES
• CALCULATE SKEWNESS AT EACH HORIZON
IMPLICATIONS

• Multi-period empirical returns are more skewed than one period returns (omitting 1987 crash)
• Asymmetric volatility is needed to explain this.
• Skewness has increased since 1987, particularly for longer horizons.
• Simulated skewness is noisy because higher moments do not exist when the persistence is so close to one. Presumably this is true for the data too.
• Many other asymmetric models could be compared on this basis.