The Fiscal Roots of Inflation

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August 12, 2019

Abstract

Unexpected inflation devalues nominal government bonds. It must therefore correspond to a decline in expected future surpluses, or a rise in their discount rates, so that the real value of debt equals the present value of surpluses. I measure each component via a vector autoregression, in response to inflation, recession, monetary and fiscal policy shocks. Discount rates, rather than deficits, account for much inflation and deflation. Monetary policy smooths the inflationary response to fiscal shocks. I interpret the results through a fiscal theory of monetary policy.

*Hoover Institution, Stanford University and NBER. I thank seminar participants and Harald Uhlig for helpful comments. Data, code, and updates are at http://faculty.chicagobooth.edu/john.cochrane.

1. Introduction

This paper measures the fiscal roots of inflation. I start with a linearized version of the government debt flow identity,

$$v_{t+1} = v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} - s_{t+1}. \tag{1}$$

The log debt to GDP ratio at the end of period $t+1$, v_{t+1}, is equal to its value at the end of period t, v_t, increased by the log real nominal return on the portfolio of government bonds r^n_{t+1} less inflation π_{t+1}, less log GDP growth g_{t+1}, and less the primary surplus[1] s_{t+1}. I derive this identity in the Appendix.

Iterating forward, we have a present value identity,

$$v_t = \sum_{j=1}^{T} s_{t+j} - \sum_{j=1}^{T} \left(r^n_{t+j} - \pi_{t+j} - g_{t+j} \right) + v_{t+T}. \tag{2}$$

The log value of government debt, divided by GDP, is the present value of future surpluses, discounted at the ex-post real return, adjusted by GDP growth.

Taking time $t+1$ innovations $\Delta E_{t+1} \equiv E_{t+1} - E_t$, taking the limit as $T \to \infty$ and rearranging, we have the unexpected inflation identity,

$$\begin{aligned} &\Delta E_{t+1}\pi_{t+1} - \Delta E_{t+1}\left(r^n_{t+1} - g_{t+1} \right) \\ &= -\sum_{j=0}^{\infty} \Delta E_{t+1} s_{t+1+j} + \sum_{j=1}^{\infty} \Delta E_{t+1}\left(r^n_{t+1+j} - \pi_{t+1+j} - g_{t+1+j} \right). \end{aligned} \tag{3}$$

A decline in the present value of surpluses, coming either from a change in surpluses or a rise in their discount rates, must result in a lower real value of the debt. This reduction can come about by unexpected inflation, or by a decline in nominal long-term bond prices. I focus on this identity because the value of the debt v_t drops out.

(I approximate around $r = g$ so there is no discounting in the above sums. One can also approximate around $r > g$ to allow $\beta^j = e^{j(r-g)}$ terms in the sums. However, the variables are all stationary, and impulse-responses converge faster than β^j, so expected values converge, and downweighting higher order terms by something like 0.99^j makes little difference to the results. Since the value of the debt v_t is stationary, $\lim_{T\to\infty} \Delta E_{t+1} v_{t+T} = 0$. One can also view these

[1] More precisely, s_{t+1} is the real primary surplus divided by GDP, and scaled by the steady state debt-to-GDP ratio, so its units are real surplus divided by real value. It can also represent the real primary surplus divided by the previous period's real value of debt – either definition leads to the same linearization. For brevity, I refer to s_{t+1} as simply the "surplus." I impute the surplus from the other terms of (1), so its identity really only matters when one wishes to assess the accuracy of approximation, which I do below, or to assess an independent data source on surpluses.

unweighted identities as $r \to g$ limits.)

I use a vector autoregression (VAR) to measure each component of the unexpected inflation identity (3), in response to a variety of shocks. The main message is that discount rates matter. Unexpected inflation corresponds more to a rise in the discount rate for government debt than to a decline in primary surpluses.

The second term of (3) is also a key point. For example, when there is a negative innovation to the present value of surpluses on the right hand side, a decline in nominal long-term bond prices and consequent negative return $\Delta E_{t+1} r^n_{t+1}$ can lower the real value of debt, in place of unexpected inflation $\Delta E_{t+1}\pi_{t+1}$. In this way, long-term debt can buffer fiscal shocks.

To evaluate this channel, I examine the size of responses $\Delta E_{t+1} r^n_{t+1}$, and I break them down to expected future inflation vs. expected future real returns. With a geometric maturity structure, in which the face value of maturity j debt declines at rate ω^j, the Appendix develops the approximate identity

$$\Delta E_{t+1} r^n_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1}\left[(r^n_{t+1+j} - \pi_{t+1+j}) + \pi_{t+1+j}\right]. \tag{4}$$

Lower bond prices correspond to higher bond expected nominal returns, which in turn are composed of real returns and inflation. I find that the bond return responses $\Delta E_{t+1} r^n_{t+1}$ are large, and that they mostly correspond to changes in expected future inflation, not to changes in expected real returns. As a result, the contemporaneous bond return term $\Delta E_{t+1} r^n_{t+1}$ can show us how a fiscal shock is absorbed by a shock to long-term expected inflation, which slowly drains repayment of long-term bonds, rather than a price level jump. In this way expected and unexpected inflation are connected.

Overall, then, I find two novel descriptions of the fiscal roots of inflation: Inflationary and deflationary fiscal shocks come to a large extent from discount rate variation, not from shocks to expected surpluses, and fiscal shocks translate via long-term debt to persistent movements in expected inflation. Strong forecastable surplus movements or sudden debt-devaluing price level jumps are not the only, or dominant, fiscal roots of inflation.

I interpret the results through the lens of the fiscal theory of monetary policy, which I briefly define, summarize and extend. In this interpretation, unexpected inflation is *caused* by movements of the other terms of the identity. We study the fiscal roots rather than the fiscal consequences of inflation. In this view, the point of the paper is to establish a set of facts for constructing such models, much as atheoretical VARs guided the construction of conventional monetary models.

But the above identities hold in almost all macroeconomic models used to quantitatively address inflation. (They assume that the present value is finite – loosely that $r > g$ – ruling out models of dynamic inefficiency. I presume this case without further comment.) Likewise, the VARs impose no theory. Therefore, the results can also be read as measuring the nature of passive-fiscal adjustments to an active-money (Leeper (1991)) regime, if one so wishes. A well-specified active-money regime must spell out a realistic passive-fiscal policy. The fact that discount rates do much of the adjusting, rather than the painless ex-post lump-sum taxes alluded to in many theoretical footnotes, changes the fiscal underpinnings of such models substantially.

Since the analysis is based on identities that hold in both sets of models under consideration, I do not test anything. But which element in an identity moves is still an interesting measurement, for any model.

1.1. Literature

Much of the technique in this paper is imported from asset pricing. The general approach to linearizing the valuation identity follows Campbell and Shiller (1988). The Appendix relates impulse-response calculations to asset pricing variance decompositions. The summary in Cochrane (2011b) and treatment of identities in Cochrane (2007) are obvious precursors. The uniting theme in the former is that asset price and return variation is largely driven by variation in discount rates. Using analogous techniques, this paper finds the same result in questions of government debt and inflation, where discount rate variation is usually ignored.

The analysis of government finances, how debt is paid off, grown out of, or inflated away, is a huge literature. Hall and Sargent (1997), Hall and Sargent (2011) are the most important precursors. Hall and Sargent focus on the market value of debt, as I do, not the face value reported by the Treasury, and consequent proper accounting for interest costs. I use data provided by Hall, Payne, and Sargent (2018).

This paper uses the innovation identity (3), to focus on inflation, paralleling the return decompositions from asset pricing. A companion paper Cochrane (2019) decomposes the value of government debt v_t, starting from the value identity (2), paralleling the price/dividend ratio decompositions from asset pricing. Since inflation turns out to be an insignificant part of that story, the two exercises make largely orthogonal points, despite their common methodological heritage.

The fiscal theory of monetary policy is the latest step in a long literature on the fiscal theory of the price level. Sims (2011) and Cochrane (2017) are immediate antecedents. Much of the fiscal theory literature has pursued various theoretical controversies. A big point of this paper is to begin productively use fiscal theory to understand US data.

2. The fiscal theory of monetary policy

By the term "fiscal theory of monetary policy," I mean a model that uses the new-Keynesian/DSGE ingredients, including a central bank that follows an interest rate target, but substitutes active fiscal for active monetary policy to select equilibria, following the fiscal theory of the price level.

Here I set forth a few simple models of this sort, that express the mechanisms I use later to interpret of empirical results. The resulting picture differs substantially from the usual impression of the fiscal theory of the price level. Central banks, and their interest rate targets remain centrally important for the determination of expected and unexpected inflation. The models capture persistent inflation, not just large price level jumps. There is a strong role for discount rates, and no prediction of easily-refutable correlations between inflations and debt or deficits.

2.1. Simplest FTMP model, and active/passive assumptions

As the simplest fiscal theory of monetary policy example, consider a frictionless model composed of only the Fisher equation (linearized intertemporal first-order condition in a constant-endowment economy) with a constant real interest rate, and flexible prices:

$$i_t = r + E_t \pi_{t+1}. \tag{5}$$

With no growth and one-period debt, the fiscal inflation identity (3) reduces to

$$\Delta E_{t+1} \pi_{t+1} = -\Delta E_{t+1} \sum_{j=0}^{\infty} s_{t+1+j}. \tag{6}$$

(With one-period debt, the return on government bonds is $r^n_{t+1} = i_t$.)

If the central bank sets an interest rate target $\{i_t\}$, and with a passive fiscal policy in which surpluses s_t react ex-post to make (6) hold for any inflation rate, this model determines expected inflation but not unexpected inflation. There are multiple equilibria corresponding to any value of unexpected inflation $\Delta E_{t+1}\pi_{t+1}$.

The standard new-Keynesian approach solves this multiplicity by specifying an active monetary policy

$$(i_t - i_t^*) = \phi \left(\pi_t - \pi_t^*\right), \phi > 1, \tag{7}$$

where π_t^*, i_t^* are equilibrium values the central bank wishes to select from the multiple equilibria of (5). One also adds a rule against nominal explosions $\lim_{T\to\infty} E_{t+1}\pi_{t+T} = 0$. Now only one value of unexpected inflation remains. (Woodford (2003), Cochrane (2011a).)

For example, suppose the central bank wishes to produce an AR(1) inflation process,

$$\pi^*_{t+1} = \theta\pi^*_t + \varepsilon_{t+1}. \tag{8}$$

By (5), the equilibrium interest rate must follow

$$i^*_t = r + \theta\pi^*_t. \tag{9}$$

The central bank cannot simply set a time-varying peg following (9), however, as this specification would not determine unexpected inflation $\Delta E_{t+1}\pi_{t+1} = \varepsilon_{t+1}$. The central bank also specifies $\phi > 1$ and announces (7), that should another inflation $\pi_{t+1} \neq \pi^*_{t+1}$ emerge, the central bank will lead the economy to hyperinflation or deflation. The latter provision and the rule against nominally explosive equilibria selects (8) as the unique equilibrium.

Equation (7) is more commonly written

$$i_t = r + \phi\pi_t + v_t, \tag{10}$$

with $v_t = i^*_t - r - \phi\pi^*_t$. I write it in the equivalent form (7) to emphasize that "monetary policy," the the interest rate rule (9) that we observe in equilibrium, is separate from "equilibrium selection policy," the threat (7), unobserved in equilibrium, that the central bank uses to select one of multiple equilibria.

A fiscal theory of monetary policy specifies instead a "passive" $\phi < 1$ equilibrium-selection policy (7) – or, really no such policy at all, $\phi = 0$, just erasing (7) – and turns off the passive fiscal assumption, so (6) does not hold automatically. One need not assume that surpluses are exogenous. Surpluses may react endogenously to other variables. The only restriction is that surpluses do not react one-for-one to multiple-equilibrium unexpected inflation, in such a way that (6) holds for any unexpected inflation. Now the combination (5) and (6) uniquely determine both expected and unexpected inflation.

Central banks remain powerful in this model. Central banks cannot directly affect fiscal policy, and can no longer count on a "passive" fiscal adjustment to validate their equilibrium-selection policy. But central banks still set interest rate targets[2], and thereby they control expected inflation via (5). Interest rate targets may also follow rules and react endogenously to eco-

[2] *How* does the central bank set an interest rate target? Even in this cashless and frictionless model, the central bank can set interest rates by varying the quantity of debt, without changing surpluses. Briefly, for example, writing

nomic variables as in (9). Fiscal events only determine unexpected inflation, the instantaneous response of inflation to a shock $\Delta E_{t+1}\pi_{t+1}$. Monetary policy determines the rest, $\Delta E_{t+1}\pi_{t+1+j}$, $j > 0$.

This frictionless model is Fisherian: A rise in interest rates, with no change in surpluses, produces a rise in expected inflation one period later via $i_t = E_t\pi_{t+1}$, and no change in current inflation $\Delta E_{t+1}\pi_{t+1} = 0$. The frictionless new-Keynesian counterpart with $\phi > 1$ can produce a negative inflation response, by selecting instead an equilibrium with $\Delta E_{t+1}\pi_{t+1} < 0$. It implicitly assumes a coincident fiscal contraction, as (6) still holds, achieved by "passive" fiscal authorities. The fiscal theory of monetary policy can produce the same response, but would call it a coordinated fiscal and monetary shock. In the next sections, we see how a negative inflation response can emerge even without a shock to surpluses.

2.2. Long-term debt

Now, add long-term debt with a geometric maturity structure, keeping for now a constant real interest rate and flexible prices. The inflation (3) and bond return identities (4) specialize to

$$\Delta E_{t+1}\pi_{t+1} - \Delta E_{t+1}r^n_{t+1} = -\sum_{j=0}^{\infty}\Delta E_{t+1}s_{t+1+j} \tag{11}$$

$$\Delta E_{t+1}r^n_{t+1} = -\sum_{j=1}^{\infty}\omega^j\Delta E_{t+1}\pi_{t+1+j}. \tag{12}$$

Substituting the return identity into the inflation identity, the model simplifies to the Fisher equation (5), $i_t = E_t\pi_{t+1}$, and

$$\sum_{j=0}^{\infty}\omega^j\Delta E_{t+1}\pi_{t+1+j} = -\sum_{j=0}^{\infty}\Delta E_{t+1}s_{t+1+j} \tag{13}$$

in place of (6). With long-term debt, shocks to the present value of surpluses in (13) correspond to a change in the weighted average of current and, now, expected future inflation.

When surpluses do not move, (13) introduces an important link between changes in ex-

the nonlinear valuation identity with a constant interest rate as

$$\frac{B_{t-1}}{P_t} = E_t\sum_{j=0}^{\infty}\beta^j s_{t+j},$$

then a change in B_{t-1} at time $t-1$ with no change in surpluses changes expected inflation $E_{t-1}(P_{t-1}/P_t)$ and therefore the nominal interest rate. See Cochrane (2017) Section 2.4 for an extended discussion. Alternatively, one may appeal to the standard Woodford (2003) cashless limit argument.

pected inflation at different dates. Consider a persistent monetary policy shock – a persistent positive change in $i_t = E_t \pi_{t+1}$ with no change in surpluses. From (13), if the terms $\Delta E_{t+1}\pi_{t+1+j}$ for $j \geq 1$ are all positive, then $\Delta E_{t+1}\pi_{t+1} < 0$. In this way, with long-term debt, this positive persistent monetary policy shock induces a negative initial inflation response, which briefly overturns the otherwise Fisherian properties of this frictionless model. Inflation here behaves much the same way that bond yields do: a rise in expected future inflation leads to a decline in current inflation. This observation boils down a large effort in more complex models to produce a negative inflation response in a rational expectations model, without assuming a contemporary fiscal contraction (Sims (2011), Cochrane (2017), Cochrane (2018).)

For example, suppose the central bank creates a monetary disturbance that follows an AR(1),

$$i_t = \rho i_{t-1} + \varepsilon_t^i, \tag{14}$$

with no change in surpluses. The inflation response to a unit shock $\varepsilon_1^i = 1$ is higher expected inflation for all periods $2, 3, ...,$

$$\Delta E_1 \pi_{1+j} = \Delta E_1 i_j = \rho^j.$$

However, from (13), the impact effect of a higher interest rate is negative:

$$\Delta E_1 \pi_1 = -\frac{\rho\omega}{1-\rho\omega}.$$

This is an example, not a theorem. Sticky prices in the next section change the dynamics. Also, if the discount rate rises, that lowers the present value of surpluses, an inflationary force that can offset or even overcome this simple mechanism. The point here is to understand one of several mechanisms that help us to interpret the empirical results below.

In (13), with short-term debt $\omega^j = 0$, a fiscal shock must result in an immediate inflation. A price-level jump is the only way to devalue short-term debt. With long-term debt, a rise in expected future inflation can devalue long-term bonds. Thus a fiscal shock may give rise to a persistent small rise in inflation, or even a rise only in future expected inflation with no current change at all. As we shorten the time interval, the effective ω rises, and the instantaneous term drops out altogether; all fiscal shocks now correspond to changes in expected future inflation.

Since monetary policy controls the interest rate, monetary policy determines whether a fiscal shock is felt entirely in current inflation $\Delta E_{t+1}\pi_{t+1}$, or leads to a drawn-out but smaller inflation $\Delta E_{t+1}\pi_{t+j}$.

Contrary to the impression one gets with short-term debt models, then, fiscal theory does not produce only one-period price level shocks. A persistent inflation following a fiscal shock, if it is accommodated by monetary policy, can emerge from this model as well. This observations suggests a story for the 1970s.

In these ways and more, the addition of the ω^j terms in (13) represents a major shift in perspective. Equation (13) alerts us to think of the innovation in current and expected future inflation, captured in the bond return, together, and not to focus on the innovation to current inflation alone. I do not substitute the bond return identity (4) into the general inflation identity (3) because the result is not pretty, I don't wish to impose a geometric maturity structure, and because variation in bond expected returns and discount rates cloud the picture. But we will see this basic mechanism at work in the estimates below.

2.3. Sticky prices and policy rules

Finally, let us examine the simplest fiscal theory of monetary policy model with sticky prices as well as long-term debt. I use the standard new-Keynesian intertemporal substitution and Phillips curves,

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) \tag{15}$$

$$\pi_t = \beta E_t \pi_{t+1} - \kappa x_t. \tag{16}$$

Eliminating the output gap x_t from (15)-(16), we have

$$\beta E_t \pi_{t+2} - (1 + \beta - \sigma\kappa) E_t \pi_{t+1} + \pi_t = \sigma\kappa i_t. \tag{17}$$

We can write this equation that expected inflation $E_t \pi_{t+1}$ is a two-sided exponentially-weighted moving average of the interest rate i_t, with weights given by the roots of the lag polynomial (17) (Cochrane (2018) p. 165), naturally generalizing the Fisher equation (5) $E_t \pi_{t+1} = i_t$. Therefore, monetary policy can still determine expected inflation. It just takes a more complex interest rate path to give any particular expected inflation path.

We can compute responses to specified interest rate and surplus paths as before, but it is more interesting to specify policy rules. I also thereby verify that one *can* specify policy rules rather than fixed surplus and interest rate paths. The resulting model can stand as a benchmark fiscal theory of monetary policy, parallel to the standard three-equation ((10),(15), (16)) new-Keynesian model.

The model consists of IS and Phillips equations (15), (16), fiscal and monetary policy rules, and the evolution of the value of government debt,

$$i_t = \theta_{i\pi}\pi_t + \theta_{ix}x_t + u_t^i \tag{18}$$

$$s_t = \theta_{s\pi}\pi_t + \theta_{sx}x_t + \alpha v_t^* + u_t^s \tag{19}$$

$$\left[1 + \left(\gamma^{-1} - 1\right)\alpha\right] v_{t+1}^* = v_t^* + i_t - E_t\pi_{t+1} - s_{t+1} \tag{20}$$

$$v_{t+1} = v_t + r_{t+1}^n - \pi_{t+1} - s_{t+1} \tag{21}$$

$$E_t r_{t+1}^n = i_t \tag{22}$$

$$r_{t+1}^n = \omega q_{t+1} - q_t \tag{23}$$

$$u_{t+1}^i = \rho^i u_t^i + \varepsilon_{t+1}^i \tag{24}$$

$$u_{t+1}^s = \rho^s u_t^s + \varepsilon_{t+1}^s \tag{25}$$

These are the same equations as the standard new-Keynesian model, with a set of implicit fiscal equations spelled out. We parameterize and solve the model with active fiscal rather than active monetary policy.

Equation (18) is the monetary policy rule. Equation (19) is the fiscal policy rule. I allow surpluses to respond to the output gap, as they do, with procyclical tax receipts and counter-cyclical stabilizers and stimuli. I allow surpluses to respond to inflation as well, which one can view either as a policy decision or imperfect indexation. I discuss v^* below. Equation (21) tracks the evolution of the real value of debt, from (1). Equation (22) is the bond pricing equation, using the expectations hypothesis that expected returns on bonds of all maturities are the same. Equation (23), derived in the Appendix, relates the return on the government bond portfolio to its price q_t. Equations (24)-(25) allow persistence in monetary and fiscal policy disturbances.

The v^* term in (19) and (20) says that the surplus responds to what the value of the debt would be with no inflation, i.e. to debts accumulated from past deficits or from changes in the ex-ante real interest rate $r_t = i_t - E_t\pi_{t+1}$. However, fiscal policy ignores changes in the value of debt that arise from unexpected and especially multiple-equilibrium inflation and deflation. This specification gives us a fiscal policy that is "active," and rules out multiple equilibria, but nonetheless pays off debts accumulated from past deficits, as the government promised to do, implicitly or explicitly, when it sold debt. Fiscal theory does not say that governments set surpluses ignoring promises made when they ran previous deficits.

This specification models a monetary-fiscal regime, such as inflation targeting or a gold standard, in which the Treasury agrees to pay its debts at the inflation target or gold standard target, but does not commit to pay larger values of its debts should (say) a large off-equilibrium deflation emerge. Like (7), this specification captures the difference between reactions observed in equilibrium and unobserved off-equilibrium responses that select equilibria.

We can also regard (19) and (20) as a way to write compactly and intuitively a dynamic surplus process in which a government running deficits promises future surpluses. It writes an otherwise complex non-invertible MA process in a VAR(1) form by introducing a latent state variable v_t^*. To see this point, consider the simple case with $\theta_{sx} = \theta_{s\pi} = 0$ and $r_t = i_t - E_t\pi_{t+1} = 0$. Substituting the surplus response (19) into the v^* process (20) we can write the v^* process as

$$v_{t+1}^* = -\frac{\left(1+\frac{\alpha}{\gamma}\right)^{-1}}{1-\left(1+\frac{\alpha}{\gamma}\right)^{-1}L}u_{t+1}^s.$$

Substituting this result back into (19) we obtain a surplus process

$$\begin{aligned} s_{t+1} = a(L)u_{t+1}^s &= \left[1-\alpha\frac{\left(1+\frac{\alpha}{\gamma}\right)^{-1}}{1-\left(1+\frac{\alpha}{\gamma}\right)^{-1}L}\right]u_{t+1}^s \\ &= u_{t+1}^s - \frac{\alpha}{\left(1+\frac{\alpha}{\gamma}\right)}u_{t+1}^s - \frac{\alpha}{\left(1+\frac{\alpha}{\gamma}\right)^2}u_t^s - \frac{\alpha}{\left(1+\frac{\alpha}{\gamma}\right)^3}u_{t-1}^s - \ldots \end{aligned}$$

A deficit shock, a negative u_t^s, is followed by a string of small positive surpluses, which pay back the debt, or some of it.

The total response of the surplus to a shock is

$$a(1) = 1-\gamma.$$

With $\gamma = 1$, the government pays back all of the deficit, there are no shocks to the expected sum of future surpluses. The parameter $\gamma < 1$ allows us to model a surplus process in which the government promises to pay back part of the deficit rather than all of the deficit, letting unexpected inflation soak up the rest by devaluing outstanding bonds. Intuitively, with $\gamma < 1$, (20) describes what the value of the debt would be if someone else came along with a surplus that pays $\left(\gamma^{-1}-1\right)\alpha$ of the debt.

Like the rest of the model, this surplus process can and should be generalized towards realism in many ways. In particular, the v^* process can respond to one particular value of unexpected inflation, rather than the strict zero-inflation target here. News about future surpluses and historical episodes are likely not well modeled by AR(1) shocks.

The IS and Phillips curves (15)-(16) leave two undetermined expectational errors, needing two forward-looking roots to give a unique equilibrium. As usual, they have one forward and one backward-looking root, so we need one extra forward-looking root. In active-money new-Keynesian models $\theta_{\pi i} > 1$ (roughly speaking) generates the additional explosive root. I specify passive monetary policy with $\theta_{\pi i} < 1$. (With more complex specifications, one can create a passive-money model in which regressions of interest rates on inflation have a coefficient greater than one. As this model is not elaborated to be empirically realistic in its other equations, especially the IS and Phillips curves (15)-(16), I do not pursue that complication here.)

With short-term debt, we would have $r^n_{t+1} = i_t$, and the combination (19)-(21) would provide the extra explosive or unit root. Long-term debt adds another expectational error, (22), but one more unstable root in (23). Together (22)-(23) solve forward to

$$q_t = -E_t \sum_{j=1}^{\infty} \omega^{j-1} i_{t+j-1},$$

the expectations hypothesis that long-term bond prices reflect an average of future short-term interest rates.

The Appendix documents the algebra for solving the model in the standard way.

2.4. Responses

The top panel of Figure 1 presents the response of this model to an AR(1) fiscal policy disturbance u^s_t, in the case of no policy responses to endogenous variables $\theta = 0$. The interest rate i and long-term bond return r^n do not respond in this case. Inflation rises and decays with an AR(1) pattern, not the one-period price-level jump that the frictionless model produces in this case. Output rises mirroring the path of inflation, following the forward-looking Phillips curve that output is high when inflation is greater than future inflation.

We can see here several aspects of the surplus process (19) - (20) at work. The surplus s_t and the AR(1) surplus disturbance u^s_t are not the same. The surplus initially declines, but these deficits raise the v^* latent variable, which accumulates past surpluses. A long string of small positive surplus responses on the right side of the graph then partially repays the incurred debt.

This graph warns us of the empirical challenges ahead, and against many apparently easy rejections of fiscal theory. It would be hard to distinguish the surplus s from the disturbance u^s in the data, as they differ only in the long run. The surplus seems to respond to the value of debt v_t though it does not do so. Such a response does not indicate passive fiscal policy.

The terms of the unexpected inflation decomposition (3) in this response function are given in the "Fiscal, no θ responses" row of Table 1. The 0.50% inflation shock corresponds to an

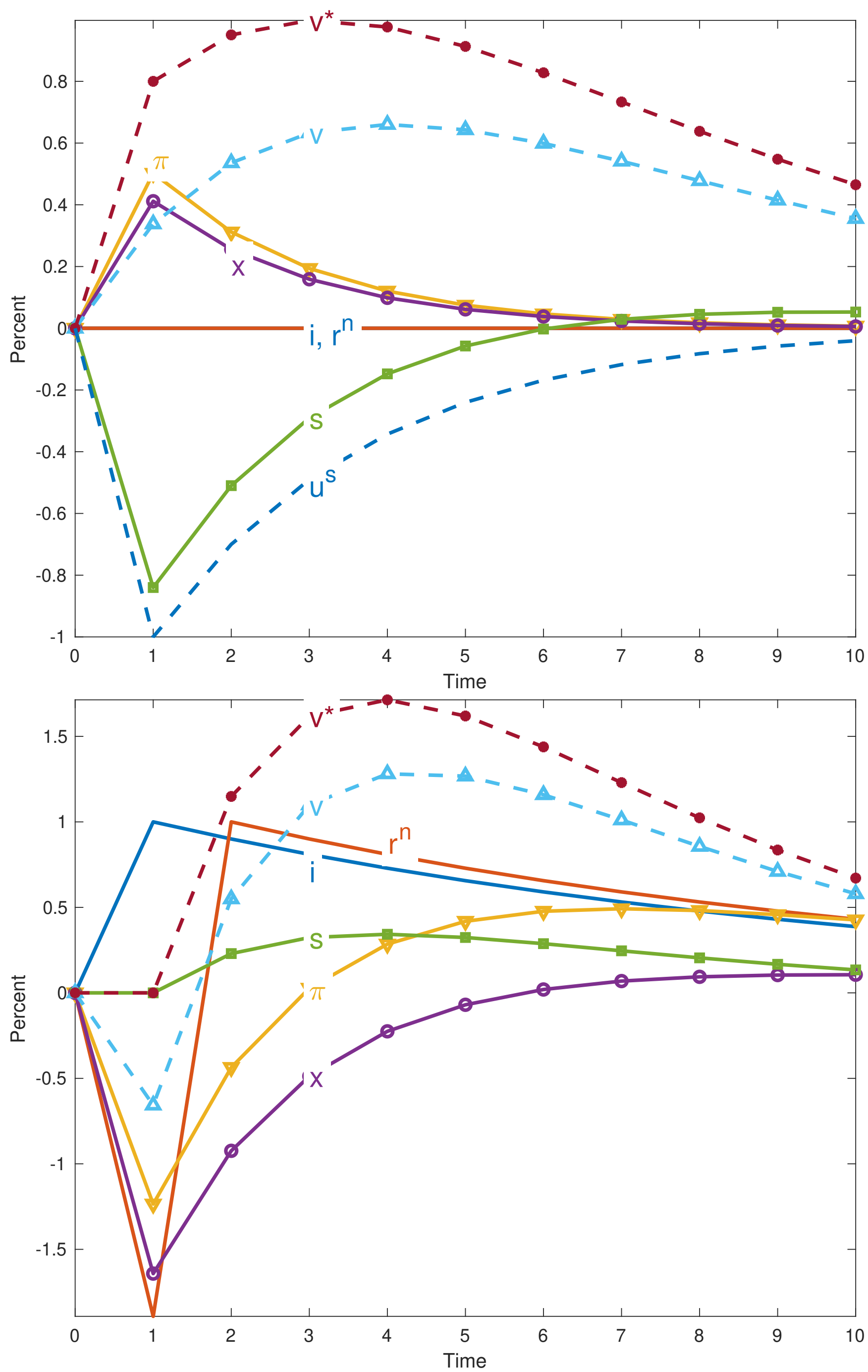


Figure 1: Responses of the sticky-price long-term debt model, with no policy responses, to AR(1) surplus shock (top panel) and AR(1) monetary policy shock (bottom panel). Parameters are $\theta = 0$, $\omega = 0.7$, $\gamma = 0.8$, $\sigma = 0.5$, $\kappa = 0.5$, $\rho^i = 0.9$, $\rho^s = 0.7$, $\alpha = 0.2$.

even larger, 1.33% decline in the sum of future surpluses. However, since there is inflation but no change in nominal rates, real rates respond negatively, which raises the value of debt and is therefore a deflationary force. This endogenous decline in discount rate buffers the effect of the surplus shock on inflation. We will see many similar offsetting responses in the VAR below.

Shock and model	$\Delta E_1 \pi_1$	$-\Delta E_1 \left(r_1^n\right)$	=	$-\sum_{j=0}^{\infty} \Delta E_1 s_{1+j}$	$+\sum_{j=1}^{\infty} \Delta E_1 \left(r_{1+j}^n - \pi_{1+j}\right)$
Fiscal, no θ responses	(0.50)	$-(0.00)$	=	$-(-1.33)$	$+(-0.82)$
Fiscal, yes θ responses	(0.23)	$-(-0.34)$	=	$-(-0.72)$	$+(0.14)$
Monetary, no θ responses	(-1.24)	$-(-1.89)$	=	$-(2.63)$	$+(3.28)$
Monetary, yes θ responses	(-0.60)	$-(-0.80)$	=	$-(1.14)$	$+(1.35)$

Shock and model	$\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j}$	=	$-\sum_{j=0}^{\infty} \Delta E_1 s_{1+j}$	$+\sum_{j=1}^{\infty} (1-\omega^j) \Delta E_1 \left(r_{1+j}^n - \pi_{1+j}\right)$
Fiscal, no θ responses	(0.89)	=	$-(-1.33)$	$+(-0.44)$
Fiscal, yes θ responses	(0.60)	=	$-(-0.72)$	$+(-0.11)$
Monetary, no θ responses	(-1.08)	=	$-(2.63)$	$+(1.55)$
Monetary, yes θ responses	(-0.29)	=	$-(1.14)$	$+(0.84)$

Table 1: Terms of the inflation decomposition for sticky-price model response functions.

In the absence of growth, we can productively substitute the return identity (4) into the inflation identity (3) here to obtain

$$\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j} = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} + \sum_{j=1}^{\infty} (1-\omega^j) \Delta E_1 \left(r_{1+j}^n - \pi_{1+j}\right). \tag{26}$$

As above, with long term debt, fiscal shocks spread over current and future inflation, weighted by the maturity structure of the debt, and with the time path of inflation dependent on the monetary policy response. Like (13), this expression emphasizes that the separation between unexpected inflation at time 1 and changes in expected inflation at later periods are not separate phenomena, or equivalently that the period one bond return shock $\Delta E_1 r_1^n$, is not an independent phenomenon. Fiscal pressures affect inflation at all dates, marked to market in the bond return. Relative to (13), this expression adds discount rate variation. Since $\omega < 1$, a change in discount rate does not cancel from bond returns and the inflation identity. For example, a permanent rise in discount rate lowers the present value of surpluses more than it lowers the value of outstanding government bonds.

Table 1 includes this decomposition as well. The sum of surpluses is the same. In this case, the overall weighted inflation response of 0.89% comes from the -1.33% cumulative surplus decline, moderated by the offsetting -0.44% weighted discount rate effect.

The cumulative surplus disturbance $\Delta E_1 \sum_{j=0}^{\infty} u^s_{1+j} = -3.33\%$ in this case is much larger than the decline in surpluses $\Delta E_1 \sum_{j=0}^{\infty} s_{1+j} = -1.33\%$. The difference comes from the partial repayment promise, the response of surplus to v^*_t, and the parameter $\gamma = 0.8$. For $\gamma = 0$, surpluses would follow $s_t = u^s_t$ and we would have a 3.33% inflation. For $\gamma = 1$, we would have larger positive surpluses on the right hand side of the graph, $\Delta E_1 \sum_{j=0}^{\infty} s_{1+j} = 0$ and no inflation.

The bottom panel of Figure 1 presents the response of variables in this model to an AR(1) monetary policy shock u^i_t, again with no responses to endogenous variables $\theta = 0$. Inflation π now declines persistently, not just for one period, in contrast to the flexible-price model of the last section. Output also declines, following the new-Keynesian Phillips curve. The yield on long-term bonds (not shown) rises, following the expectations hypothesis. This rise in yield results in a sharp unexpected negative nominal bond return r^n. Then expected bond returns rise, following the interest rate rise and the expectations model. Subtracting inflation from these nominal bond returns, the expected real rate and real bond return rises.

Surpluses are not constant. Here, I define a monetary policy shock that holds constant the fiscal policy *disturbance* $u^s_t = 0$, but not surpluses s_t themselves. Surpluses respond to the increased value of the debt v^* that results from higher real interest rates.

The terms of the unexpected inflation decomposition (3) in this response function are given by the "Monetary, no θ responses" row of Table 1. The -1.24% unexpected disinflation comes from a balance of competing forces. There is now a large, 2.63% rise in subsequent surpluses, which on their own would give rise to -2.63% unexpected deflation. However, higher real interest rates add an even larger, 3.28%, *in*flationary discount-rate effect, so that overall the present value of surpluses rises by 3.28 - 2.63 = 0.65%. But the persistent rise in nominal rates implies a -1.89% decline in the nominal value of long-term bonds, which soaks up more than all that inflation. The decomposition in the bottom panel says that the -1.08% decline in weighted inflation comes from the same 2.3% rise in surpluses, offset by an 0.84% rise in discount rate.

Discount rates matter, the path of expected inflation, interest rates, and expected bond returns matters, and even in this simple example accounting for inflation involves multiple, and often countervailing fiscal forces. Do not be surprised to see the same thing in the data to follow.

Figure 2 plots responses to the same fiscal and monetary shocks, but now adds endogenous policy responses, modifying (18)-(19) to

$$i_t = 0.5\,\pi_t + 1.0\,x_t + u^i_t \tag{27}$$

$$s_t = 0\,\pi_t + 0.5\,x_t + 0.2\,v^*_t + u^s_t. \tag{28}$$

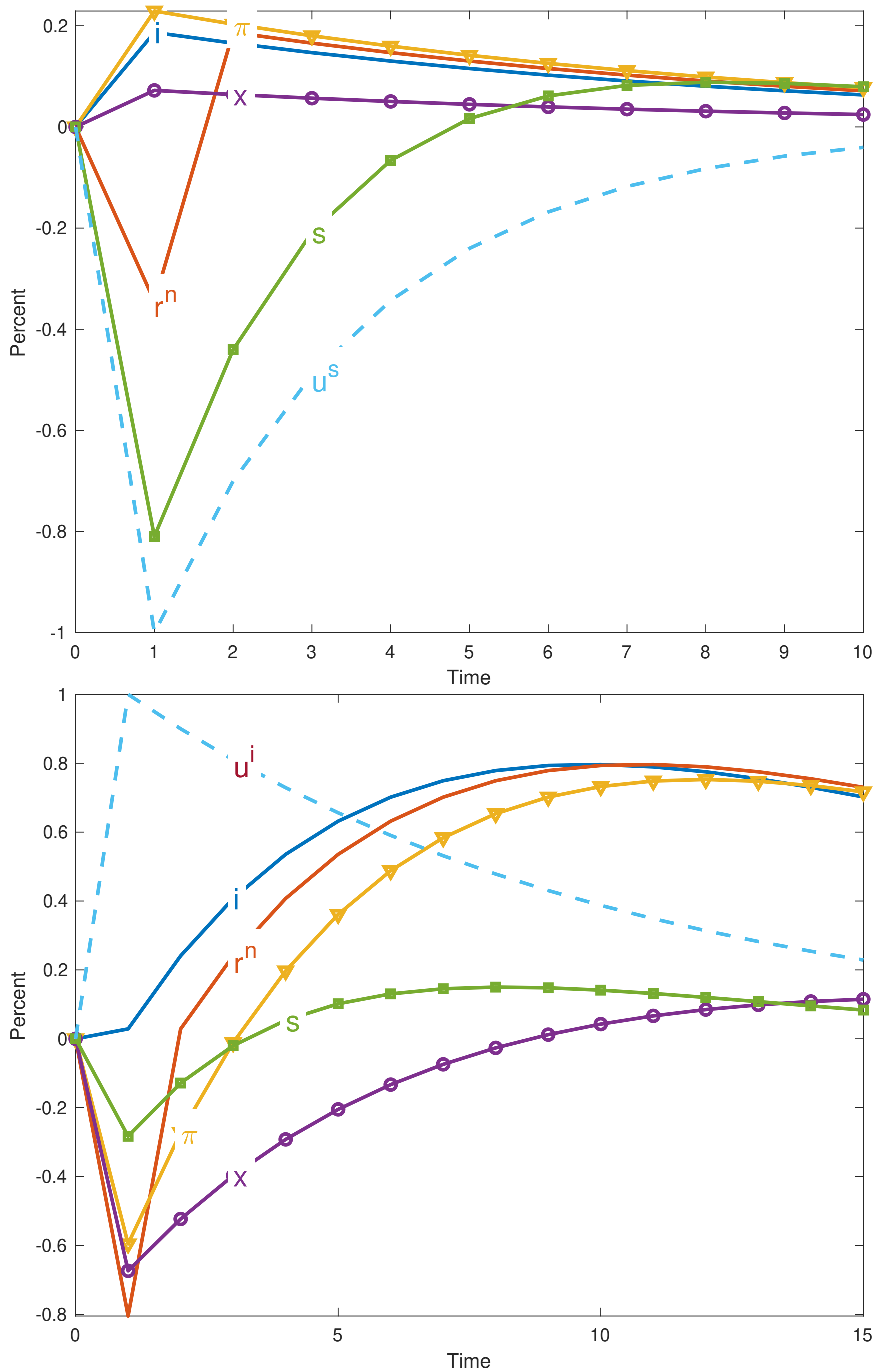


Figure 2: Response to a fiscal (top) and monetary (bottom) policy shock in the sticky-price long-term debt model, with endogenous policy responses. Parameters add $\theta_{ix} = 1$, $\theta_{i\pi} = 0.5$, $\theta_{sx} = 0.5$, $\theta_{s\pi} = 0$.

I choose the coefficients to make the graphs clear, not in an attempt at realism. Table 1 includes the inflation decompositions, in the rows marked "yes θ responses."

In the bottom monetary policy response of Figure 2 we see that the interest rate i is no longer the same as the disturbance u^i. The positive response of the interest rate target to inflation and output, and the decline in inflation and output after the shock, pull down the actual interest rate from its disturbance. I held down the coefficient $\theta_{i\pi} = 0.5$, rather than a more traditional larger value, to keep the interest rate response from being negative, the opposite of the shock. Interest rates that go in the opposite direction from monetary policy shocks are a common feature in new–Keynesian models of this sort. (Cochrane (2018) p. 175 shows some examples.) But they are confusing, and my point here is to illustrate mechanisms. The interest rate then rises gradually, along with inflation, before settling down long past the right end shown in the figure. Long-term bonds again suffer a negative return on impact, as the yield rises, and then follow the interest rate as before.

The surplus behaves quite differently than before. In Figure 1, the surplus rises, reacting only to the larger value of debt coming from a positive expected return. Now, the output decline following the monetary policy shock leads to a large deficit. The surplus eventually rises to partially, but not totally, pay off that debt.

Just how one defines and orthogonalizes monetary and fiscal policy is a subtle matter. Here I define a monetary policy shock that does not affect the fiscal shock u_t^s. But monetary policy nonetheless has fiscal consequences: The fiscal rule responds to output, (potentially) to inflation, and to real-interest-rate-induced rises in the value of debt. This is not "passive" fiscal policy in the traditional definition, since it does not respond to multiple-equilibrium unexpected-inflation induced variation in the value of the debt. But nonetheless these are sensible fiscal consequences of monetary policy which an analysis of "what if we raise interest rates and the Treasury behaves as normal" should – and, the central point here, can – include. There is no right or wrong here, there is only more or less interesting. Different definitions and orthogonalizations just ask different policy questions.

The inflation decomposition in the "Monetary, yes θ responses" rows of Table 1 show that the overall surplus response is still positive though half of its value without endogenous monetary and fiscal policy responses. Surpluses still respond positively to the higher real government debt returns, and thus are still a deflationary force. In the top decomposition, the smoother interest rate path implies a lower return shock $\Delta E_1 r_1^n$, and the discount rate effect is also less than half its previous value. In the bottom decomposition, since disinflation was temporary, the weighted sum of inflation is much smaller than the first period's inflation, -0.29% rather than

-0.60%. It corresponds to the same 1.14% rise in surpluses, mitigated by a 0.84% rise in weighted discount rate.

The top panel of Figure 2 presents the response to a fiscal shock, holding constant the monetary policy disturbance u_t^i but now allowing a monetary policy rule. The instantaneous inflation shown in Table 1 is about half its value with monetary and fiscal policy rules. The fiscal shock causes inflation and output to rise. Monetary policy raises interest rates persistently in response to the inflation and output rise, and greater output gives larger fiscal surpluses. Higher nominal interest rates also occasions a fall in bond prices, which soaks up some of the fiscal shock.

This is not a realistic example, in part because of the form of the IS and Phillips curves. Fiscal shocks in the 1970s appear stagflationary, lowering output. The point is the question, not the answer: endogenous fiscal and monetary policy responses modify the economy's response to all shocks in important ways, in these models as in standard models.

2.5. The way forward

This model is still simple and unrealistic. I advance it to show what *can* be done, as well as to explain some of the mechanisms with which I interpret the VAR responses below.

One hungers, of course, for a model that one can bring to data, estimate parameters, and formally match impulse-responses to structural and policy shocks. One wishes, in the end, at least a Smets and Wouters (2007), or Christiano, Eichenbaum, and Evans (2005), adapted to fiscal theory as I adapted the textbook new-Keynesian model above, and eventually a more ambitious model incorporating the latest in financial frictions, zero bounds, and so forth. One point of the above section is that one *can* construct such models, and quite easily from a technical standpoint. But the challenges to finding the *right* model are large. My monetary policy rule is simplistic, needing at least lags and a zero bound, plus matching policy rule regressions in data. Estimating the fiscal policy rule is a challenge of similar order, not yet started, and made even more challenging by the fact that any sensible rule, such as this one, has subtle but crucial long-run responses, or a latent state variable. Then one must confront all the usual carpentry of DSGE models – the empirical troubles of the IS and Phillips curves, habits or other dynamic preferences, heterogeneity, the evident variation in risk premia over the cycle, labor market and investment frictions, and so on.

For this reason, the rest of this paper pursues an atheoretic VAR, in the Sims tradition. The purpose of the VAR is to establish a set of stylized facts on which one can begin to build such models.

3. Estimates

3.1. Data

I use data on the market value of government debt held by the public and the nominal rate of return of the government debt portfolio from Hall, Payne, and Sargent (2018). I use standard BEA data for GDP and total consumption. I use the GDP deflator to measure inflation. I use CRSP data for the three-month Treasury rate. I use the 10-year constant maturity government bond yield from 1953 on and the yield on long-term United States bonds before that date to measure a long yield.

I infer the primary surplus from the flow identities. This calculation measures how much money the government actually borrows. NIPA surplus data, though broadly similar, does not obey the flow identity. For the VAR, I infer the surplus from the linearized identity (1), at an annual frequency.

I measure the debt to GDP and surplus to GDP ratios by the ratios of debt and surplus to personal consumption expenditures, times the average consumption to GDP ratio. Debt to GDP ratios are often used to compare countries, but in our time-series application they introduce cyclical variation in GDP. We want only a detrending divisor, and an indicator of the economy's long-run level of tax revenue and spending. Potential GDP has a severe look-ahead bias. Consumption is a decent stochastic trend for GDP.

I use a data sample 1947-2018. The immense deficits of WWII would otherwise dominate the analysis, and one may well suspect that financing that war, and expectations and reality of paying it off follows a different pattern than fiscal-monetary policy in the subsequent decades of largely cyclical deficits. WWII also featured price controls, clouding inflation measurement.

To measure the accuracy of the linear approximation, I also infer the monthly real primary surplus from the exact nonlinear flow identity, Appendix equation (37). I then carry the surplus to the end of the year using the government bond return. This procedure produces an annual series for which the nonlinear flow identity (37) continues to hold in annual data.

Figure 3 presents the surplus and compares three measures. The "Linear, s_t" line imputes the surplus from the linearized flow identity (1) directly at the one-year horizon, which is the measure I use in the following analysis. The "sv_t" and "sy_t/e^v" lines both infer the surplus from the exact nonlinear flow identity (37), as above. The "sv_t" line presents the ratio of the exact surplus to the previous year's value of the debt. The "sy_t/e^v" line presents the exact surplus to GDP ratio – actually, the ratio of surplus to consumption, times the average consumption to GDP ratio – scaled by the average value to GDP ratio $e^{E(v_t)}$. The Appendix shows that linearizing

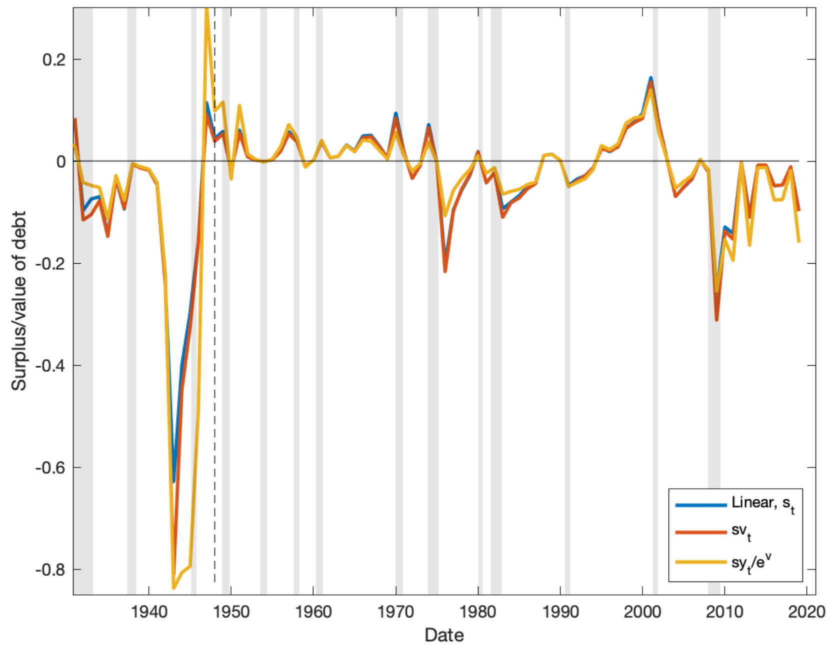


Figure 3: Surplus. "Linear" is inferred from the linearized flow identity. "sv " is the ratio of the primary surplus to the previous year's market value of the debt. "sy" is the ratio of surplus to consumption, scaled by the average consumption to GDP ratio and the average value of debt. Vertical shading denotes NBER recessions.

in terms of either concept leads to the same result, at a linearization point $r = g$. The vertical dashed line indicates the post-1947 sample that I use in VAR analysis below.

The first piece of news is that there *are* primary surpluses. One's impression of endless deficits comes from the deficit including interest payments on the debt. Even NIPA measures show regular positive primary surpluses. Steady primary surpluses from 1947 to 1975 helped to pay off WWII debt. 1975 started an era of large primary deficits, but also interrupted by the strong surpluses of the late 1990s. Postwar primary surpluses also have a clear cyclical pattern. The primary surplus correlates very well with the unemployment rate (not shown), a natural result of procyclical tax revenues, automatic (e.g. unemployment insurance) and discretionary countercyclical spending.

The three measures in Figure 3 are close. The graph is a measure of the accuracy of the linearized identity (1). The linearized identity is a slightly closer approximation to the surplus to value ratio sv. The difference is largest when the value of debt is far from its mean, both in WWII and in the 1970s.

	r^n_{t+1}	g_{t+1}	π_{t+1}	s_{t+1}	v_{t+1}	i_{t+1}	y_{t+1}
r^n_t	-0.17**	-0.02	-0.10**	-0.32*	0.28*	-0.08*	0.04*
g_t	-0.27*	0.20*	0.16*	1.37**	-2.00**	0.28**	0.06
π_t	-0.15	-0.14*	0.53**	-0.25	-0.29	0.09	0.04
s_t	0.12**	0.03	-0.03*	0.35**	-0.24*	-0.04*	-0.04**
v_t	0.01	-0.00	-0.02**	0.04*	0.98**	-0.01	-0.00
i_t	-0.32*	-0.40*	0.29*	0.50	-0.72	0.73**	0.36**
y_t	1.93**	0.54**	-0.17	-0.04	1.60*	0.11	0.46**
$100 \times std(\varepsilon_{t+1})$	2.18	1.53	1.12	4.75	6.55	1.27	0.82
R^2	0.71*	0.17*	0.73*	0.48*	0.97*	0.82*	0.90*
$100 \times std(x)$	4.08	1.68	2.16	6.61	37.00	2.96	2.63

Table 2: OLS VAR estimate. Sample 1947-2018. One (two) stars means the estimate is one (two) Monte Carlo standard errors away from zero.

3.2. Vector autoregression

Table 2 presents OLS estimates of the VAR coefficients. Each column is a separate regression. I orthogonalize shocks later, so the order of variables has no significance. The VAR includes the central variables for the inflation identity – nominal return on the government bond portfolio r^n, consumption growth rate g, inflation π, surplus s and value v. I include the three-month interest rate i and the 10 year bond yield y as they are important forecasting variables for growth, inflation, and long-term bond returns. It is important to include the value of debt v_t in the VAR, even if we are calculating terms of the innovation identity (3) that does not reference that value. When we deduce from the present value identity (2) expressions $v_t = E_t(\cdot)$, we must include v_t in the information set that takes the expectation. I use a single lag. Adding the last variable, the long-term rate, already introduces slight wiggles in the impulse-response function indicative of overfitting.

I compute standard errors from a Monte Carlo. The stars in Table 2 represent one or two standard errors above zero. Since we aren't testing anything, stars are just a visual way to show standard errors without another table.

In the first column, the long-term bond yield y_t forecasts the government bond portfolio return r^n_{t+1}. The negative coefficient on the three-month rate i_t means that the long-short spread also forecasts those returns. Since the y_t and i_t coefficients are not repeated in forecasting inflation and growth, the long rate and long-short spread forecast real, growth-adjusted, and excess returns on government bonds, as we expect from the long literature in which yield spreads forecast bond risk premia (Fama and Bliss (1987), Campbell and Shiller (1991), Cochrane and Piazzesi (2005)). The long rate y_t is thus an important state variable for measuring expected bond returns, the relevant discount rate for our present value computations.

Growth g_t is slightly persistent (0.20). The term spread $y_t - i_t$ also predicts economic growth, a common finding, and reinforcing the importance of the interest rates as state variables. Inflation π_t is persistent, with a substantial own coefficient (0.53). The interest rate also helps to predict inflation.

The surplus is somewhat persistent, with an own coefficient of 0.35. Growth g_t predicts higher surpluses, an important and realistic feedback mechanism. The surplus responds to the value of the debt, (0.04). This coefficient should not be misinterpreted to measure a passive-fiscal regime. The active vs. passive fiscal question is how surpluses respond to changes in the value of debt induced by multiple-equilibrium inflation. We cannot measure off-equilibrium responses from data drawn from equilibrium. Even a completely exogenous surplus process, in which a government borrows, then raises surpluses as promised to pay off the resulting debt, will show this coefficient, as in the example of section 2.3.

The value of the debt is very persistent, with an 0.98 own coefficient. It thus becomes the most important state variable for long-run calculations. A larger surplus s_t results in less market value of debt, v_{t+1}, (-0.24), as one expects. The long-run yield y_t forecasts a rise in the value of debt v_{t+1}, as we expect given its effect on the expected return r^n_{t+1}.

The short rate i_t is also autocorrelated with an 0.73 own coefficient. The long yield y_t does not forecast the short rate, again reflecting time-varying real returns. The long yield y is also autocorrelated, again reflecting standard yield curve dynamics.

For calculations reported below, I use the standard notation

$$x_{t+1} = Ax_t + \varepsilon_{t+1}$$

to denote the VAR.

3.3. Response to an inflation shock

I orthogonalize the inflation shock so that all other variables respond contemporaneously to the inflation shock. I specify $\varepsilon_1^\pi = 1$. Then I fill in shocks to the other variables by running regressions of their shocks on the inflation shock. For each variable z, I run

$$\varepsilon^z_{t+1} = b_{z,\pi}\varepsilon^\pi_{t+1} + \eta_{t+1}.$$

Then I start the VAR at

$$\varepsilon_1 = -\left[\begin{array}{cccccc} b_{r^n,\pi} & b_{g,\pi} & \varepsilon_1^\pi = 1 & b_{s,\pi} & \dots \end{array} \right]'.$$

This procedure is equivalent to the usual orthogonalization of the shock covariance matrix, but it is more transparent and it generalizes more easily later. I refer to the VAR innovations as the change in expectations at time 1, i.e. ΔE_1, and thus the response of variable x, j periods in the future is $\Delta E_1 x_j$.

Figure 4 plots responses to this inflation shock. The "inflation" column of Table 3 presents the terms of the decomposition (3) for impulse-response functions, i.e.

$$\Delta E_1 \pi_1 - \Delta E_1 \left(r_1^n - g_1\right) = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} + \sum_{j=1}^{\infty} \Delta E_1 \left(r_{1+j}^n - \pi_{1+j} - g_{1+j}\right). \tag{29}$$

The top panel of Figure 4 also presents the main terms in this identity. As shown in the Appendix, these terms can also be interpreted as a decomposition of the variance of unexpected inflation. Table 4 presents quantiles of the sampling distributions of the terms of the inflation decomposition, discussed below.

The inflation shock coincides with a negative surplus shock s, which builds with a hump shape. One might think these persistent deficits account for inflation. But surpluses eventually rise to pay back almost all of the incurred debt. The sum of all surplus responses is -0.06, essentially zero.

The line marked $r-g$ plots the response of the real growth-adjusted discount rate, $\Delta E_1(r_{1+j}^n - \pi_{1+j} - g_{1+j})$. These are plotted at the time of the ex-post return, $1+j$, so they are the expected return one period earlier, at time j. The line starts at time 2, where the terms of the last sum in (29) start. After two periods of no movement, this discount rate rises. The sum of all discount rate terms is 1.17%. When inflation $\Delta E_1 \pi_1$ rises 1%, more than all of the corresponding decline in the value of government debt comes from a rise in discount rates. The extra decline decline in the present value of debt shows up in bond prices. The line $r^n - g$ shows the change in the first term of (29), $\Delta E_1 \left(r_1^n - g_1\right)$, which declines by 0.23%.

In sum,

- *The decline in present value of surplus corresponding to an inflation shock comes entirely from a rise in discount rate, and not from a change in expected surpluses.*

This is an important finding for matching the fiscal theory to data, or for understanding the fiscal side of passive-fiscal models. Thinking in both contexts has focused on the presence or absence of surpluses, not the discount rate.

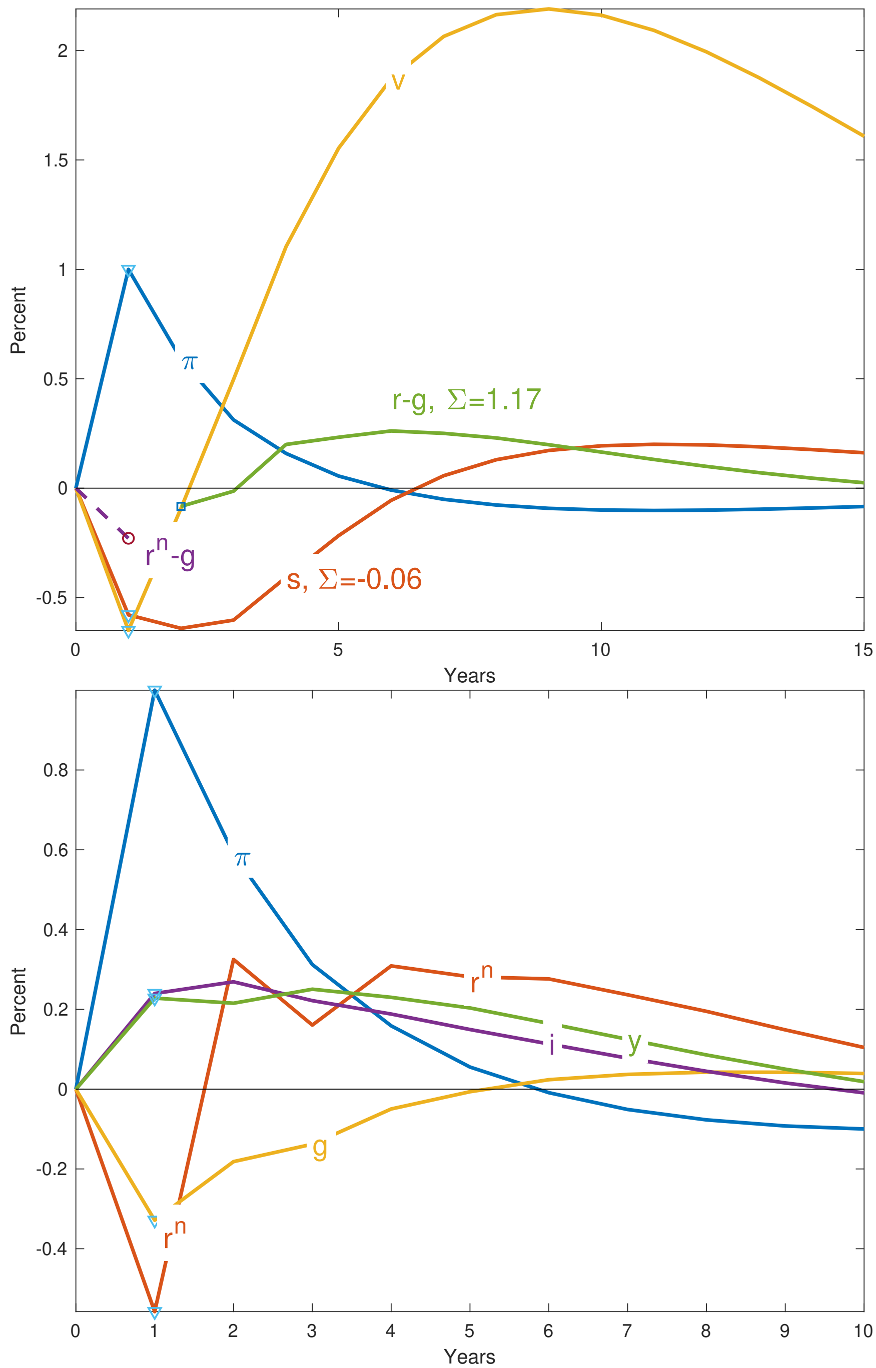


Figure 4: Response to a 1% inflation shock.

Component	Inflation	Recession	Monetary	Fiscal
Inflation π_1	1.00	-1.00	1.91	0.45
Bond return $(r_1^n - g_1)$	-0.23	2.19	-0.99	-0.04
of which r_1^n	-0.56	1.19	-0.99	0.12
of which g_1	-0.33	-1.00	-0.00	0.16
Total current $\pi_1 - (r_1^n - g_1)$	1.23	-3.19	2.90	0.50
Future $\sum s$	-0.06	-1.15	0.00	-1.00
Future $\sum(r-g)$	1.17	-4.34	2.90	-0.50
Total future $-\sum s + \sum(r-g)$	1.23	-3.19	2.90	0.50
r_1^n: future $\sum \omega^j (r_{1+j}^n - \pi_{1+j})$	-0.03	0.14	-0.96	-0.86
r_1^n: future $\sum \omega^j \pi_{1+j}$	0.59	-1.42	1.95	0.74

Table 3: Inflation identity terms. The top two panels present the terms of (29). The bottom panel presents the terms of the bond return identity (30).

	Inflation		Recession		Monetary		Fiscal	
Component	25%	75%	25%	75%	25%	75%	25%	75%
Inflation π_1	1.00	1.00	-1.00	-1.00	0.85	1.47	0.12	0.39
Bond return $(r_1^n - g_1)$	-0.45	0.00	1.96	2.40	-1.73	-0.95	-0.43	0.14
Future $\sum s$	-0.69	0.23	-1.28	0.49	-1.57	2.48	-1.00	-1.00
Future $\sum(r-g)$	0.42	1.57	-4.56	-2.60	0.00	0.00	-0.87	-0.31

Table 4: Inflation identity quantiles. 25 and 75 percent quantiles of the sampling distribution of the terms of the inflation identity, based on a Monte Carlo.

- *A fourth of the decline in present value of surpluses associated with an inflation shock is soaked up by a decline in the growth-adjusted value of long-term bonds.*

The lower panel of Figure 4 plots the response of rates of return in more detail, to give some intuition for the discount rate behavior of the upper panel, and Table 3 includes some of the relevant numerical values.

The response of growth g is negative and persistent. The inflation shock is, on average in this sample, stagflationary. Below, I isolate a shock in which unexpected inflation coincides with larger growth.

The return r_t^n takes a large one-period fall, but then rises. This is the picture of an unexpected rise in bond yields, which produces a one-period decline in bond prices but then a rise in bond expected return. (The sawtooth pattern comes from a slightly negative eigenvalue of the VAR, which is far below statistical significance.)

Both long and short bond yields rise throughout. The rise in discount rate, labeled $r - g$ in the top panel, comes mostly from the rise in nominal return with the contributions of growth and inflation largely offsetting past year 4.

The -0.23% growth-adjusted instantaneous return $\Delta E_1(r_1^n - g_1)$ in the top panel, consists of a large -0.56% negative bond return, mitigated by the negative of the -0.33% decline in growth.

This movement in long-term bond return $\Delta E_1 r_1^n$, and its consequent ability to soak up fiscal shocks, is not a separate phenomenon. By the bond return identity (4)

$$\Delta E_1 r_1^n = -\sum_{j=1}^{\infty} \omega^j \Delta E_1 \left[(r_{1+j}^n - \pi_{1+j}) + \pi_{1+j} \right], \tag{30}$$

the bond return corresponds to the change in expectations of future inflation or real returns. I choose the parameter $\omega = 0.69$ that makes identity (4) hold. The "r_1^n" rows of Table 3 then give the terms of the decomposition (4). The return $\Delta E_1 r_1^n = -0.56\%$ corresponds to $-(-0.03)\%$ expected future real rates, and $-(0.59)\%$ expected future inflation.

- *The unexpected decline in bond returns that comes with an inflation shock comes almost entirely from expected future inflation.*

The bond return could have come from variation in expected future real returns as well, requiring a more complex interpretation. In sum,

- *By maintaining a maturity structure with about three years duration, and by allowing interest rates and expected future inflation to rise when there are shocks to the present value of surpluses, the US spreads the inflationary impact of changes in the present value of surpluses forward, absorbing shocks to the present value of surpluses in long-term bond prices.*

This mechanism is not terribly important quantitatively in the point estimates so far. It is much more important in estimates that follow. Moreover, as one goes to higher frequency data, the bond return mechanism becomes more important and unexpected inflation less so.

As a reminder, the calculations do not imply or require a causal structure, nor do I make any structural claim for the shocks. The terminology "impulse-response function" can carry a misleading causal implication that we read the "responses" as the "effects" of the shock. Similarly, "shock" here just means an unexpected movement, though the terminology can suggest a more fundamental or exogenous source, and "structural" VAR exercises aim to measure such objects. The shocks here are only "innovations." In fact, my fiscal theory interpretation offers the reverse causal interpretation: The inflation shock reflects news about future surpluses and discount rates. That news in turn reflects news about future productivity, fiscal and monetary policy and other truly exogenous or structural disturbances. The statistical technique only measures the correlation between unexpected inflation and the unexpected change in other variables. For this reason, I allow contemporaneous movements in the other variables, which also may move in response to such news. Last, since we start with an identity (1) that holds ex-post, it holds ex-ante using any information set, so we do not implicitly assume that agents use only the information in the VAR in order to make the calculation. But "unexpected" here means relative to the VAR information set. Agents may see a lot more.

3.4. Recession shocks

We can use the same procedure to understand the fiscal underpinnings of other shocks. For any interesting ε_1, we can compute impulse-response functions, and thereby the terms of the decomposition (29). We can consider the calculation as a decomposition of the covariance of unexpected inflation with the shock ε_1, rather the decomposition of the variance of unexpected inflation.

I start with a recession shock. The response to the inflation shock in Figure 4 is stagflationary, in that growth falls when inflation rises. Unexpected inflation is, in this sample, negatively correlated with unexpected consumption (and also GDP) growth. The stagflationary episodes outweigh the simple Phillips curve episodes.

However, it is interesting to examine the response to disinflations which come in recessions, following a conventional Phillips curve. Such events are common, as in the recession following the 2008 financial crisis. But they pose a puzzle for the fiscal theory. In a recession, deficits soar, yet inflation declines. How is this possible? Well, it's possible and plausible that in recessions, expected future surpluses to pay off incurred debts rise along with current deficits. It's possible, though less plausible, that expected future surpluses rise even more than current deficits, raising the value of the debt and causing a disinflation. It's also possible, and more plausible, that real interest rates decline in a recession, so the discount rate for government debt declines, raising the value of debt, giving a deflationary force. Which of these channels can we see in the data?

To answer that question, we want to study a shock in which inflation and GDP go in the same direction. I simply create such as shock: I specify $\varepsilon_1^\pi = -1$, $\varepsilon_1^g = -1$. (The model is linear, so the sign doesn't matter, but the story is clearer for a recession.) Again, we want shocks to other variables to have whatever value they have, on average, conditional on the inflation and output shock. To fill out the other shocks, then, I run a multiple regression

$$\varepsilon_{t+1}^z = b_{z,\pi}\varepsilon_{t+1}^\pi + b_{z,g}\varepsilon_{t+1}^g + \eta_{t+1}$$

and I fill in the other shocks at time 1 from their predicted variables given $\varepsilon_1^\pi = -1$ and $\varepsilon_1^g = -1$. I then start the VAR at

$$\varepsilon_1 = -\left[\begin{array}{cccccc} b_{r^n,\pi} + b_{r^n,g} & \varepsilon_1^g = 1 & \varepsilon_1^\pi = 1 & b_{s,\pi} + b_{s,g} & \ldots \end{array}\right]'.$$

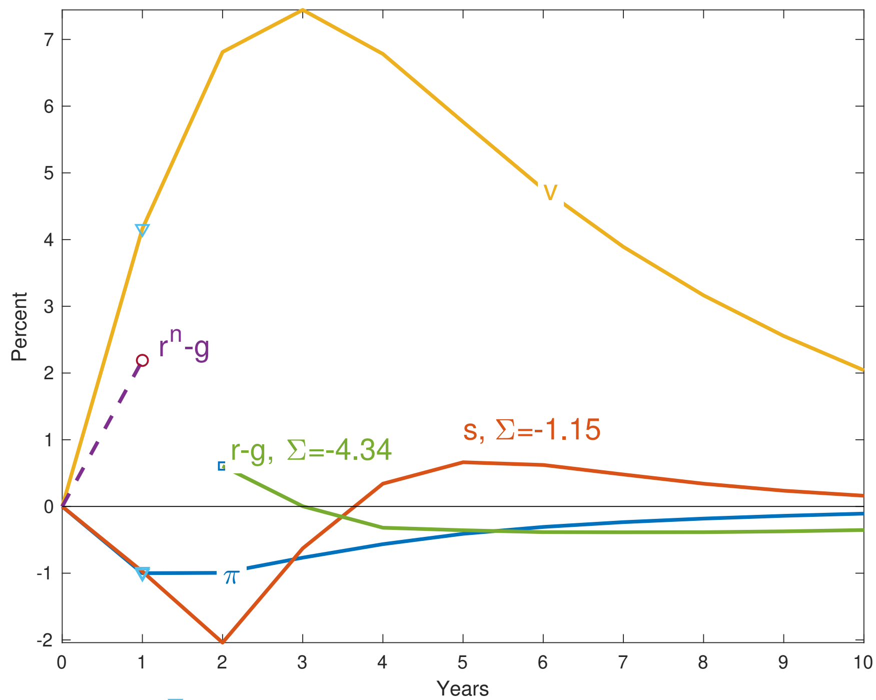
Figure 5 presents responses to this recession shock, and Table 3 collects the inflation decomposition elements in the "Recession" column.

In the bottom panel of the figure, both inflation π and growth g responses start at -1%, by construction. Consumption growth g returns rapidly, but does not much overshoot zero, so the level of consumption does not recover much at all. Consumption is roughly a random walk in response to this shock. The nominal interest rate i falls in the recession, and recovers slowly, in parallel with inflation. Long-term bond yields y also fall, but not as much as the short term rate, for about 4 years. The persistent fall in interest rate, inflation and and the smaller fall in bond yield correspond to a large positive ex-post bond return $\Delta E_1 r_1^n$. In short, we see a standard picture of a recession.

In the top panel, the recession includes a deficit s, which continues for three years. These deficits, reinforced by the positive return shock $r^n - g$ imply a large rise in the value of debt, v. These are the deficits in recessions that puzzle a simplistic interpretation of the fiscal theory. Surpluses subsequently turn positive, paying down some of the debt. But the total surplus is still -1.15%. Left to their own devices, surpluses would produce a 1.15% *inflation* during the recession. A potential story that disinflation results from future surpluses more than matching today's deficits is wrong.

Discount rates are the central story. After one period, expected real returns $r - g$ decline persistently (top panel) raising the value of debt by 4.34%. We can see the underlying forces in the bottom panel: At year 3, which are expected values at year 2, the nominal return $\Delta E_1 r_j^n$ falls more than inflation $\Delta E_1 \pi_j$, and persistently.

Even after the -1.15% decline in surplus, the 4.34% discount rate effect is larger than the -1% fall in inflation. It shows up in the current growth-adjusted bond return $\Delta E_1 \left(r_1^n - g_1\right) = 2.19\%$. That return derives from the -1% growth rate, defined in the shock, and the 1.19% positive bond return. The decomposition of bond returns in the bottom rows of Table 3 again reveals that the bond return is driven almost entirely by the persistently lower future inflation. Persistently lower future inflation would on its own lead to currently higher inflation, so we need a big fiscal shock to lower current and future inflation. So, the 4.34%-1.15% deflationary fiscal shock



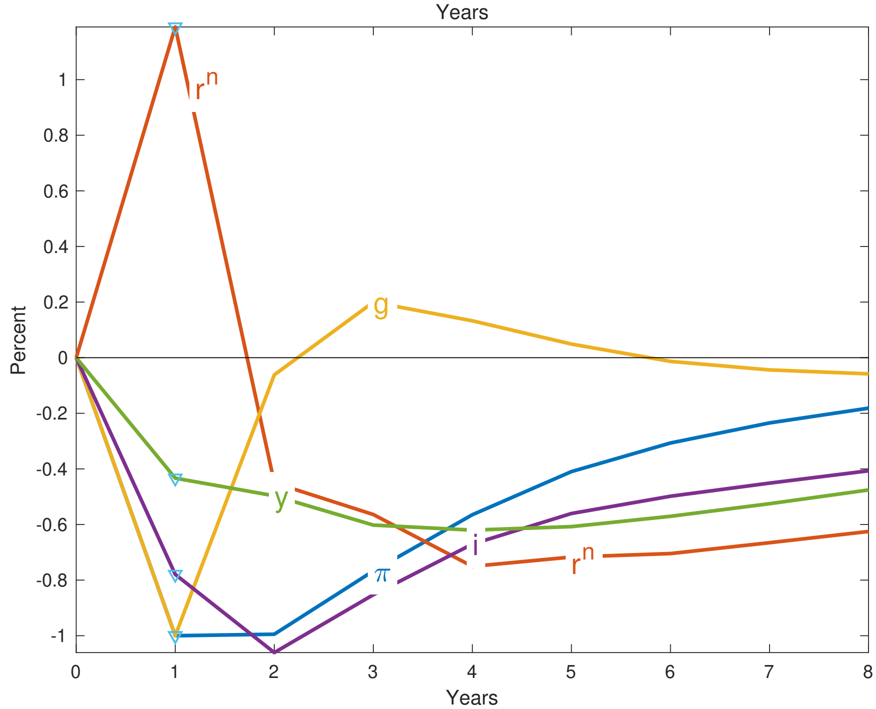


Figure 5: Response to a recession shock, $\varepsilon_1^\pi = \varepsilon_1^g = -1$.

is soaked up by the persistent disinflation.

In sum, rounding the numbers,

- *Disinflation in a recession is driven by a higher discount rate. For each 1% disinflation shock, the expected return on bonds falls so much that the present value of debt rises by 4.3%. This discount rate shock overcomes a 1.1% percent inflationary shock coming from persistent deficits, and generates persistent disinflation.*

The opposite conclusions hold of inflationary shocks in a boom. Discount rate variation gives us a fiscal Phillips curve.

3.5. Monetary and fiscal shocks

Central banks move interest rates, but cannot tax or spend. Therefore, I define here a monetary policy shock as one that moves interest rates $\Delta E_1 i_1$, but does not affect the sum of current and future surpluses, $\Delta E_1 \sum_{j=0}^{\infty} s_{1+j} = 0$. I further orthogonalize the monetary policy shock so that it does not contemporaneously move the growth rate g_t, ascribing the contemporaneous positive correlation between growth and interest rate shocks as a Taylor-rule reaction of the Fed to growth and not the other way around.

Conversely, I define here a fiscal shock as a movement in current and expected primary surpluses $\Delta E_1 \sum_{j=0}^{\infty} s_{1+j}$ that comes with no movement in the short-run interest rate $\Delta E_1 i_1 = 0$.

The response of the sum of future surpluses to a shock ε_1 is

$$\Delta E_1 \sum_{j=0}^{\infty} s_{1+j} = a_s'(I-A)^{-1}\varepsilon_1.$$

To calculate how other shocks respond instantaneously to a monetary shock, then, I run for each variable z a multiple regression

$$\varepsilon_{t+1}^z = b_{z,i}\varepsilon_{t+1}^i + b_{z,pv}a_s'(I-A)^{-1}\varepsilon_{t+1} + b_{z,g}\varepsilon_{t+1}^g + \eta_{t+1}. \tag{31}$$

The monetary policy shock wants

$$\varepsilon_1^i = 1,\ a_s'(I-A)^{-1}\varepsilon_1 = 0,\ \varepsilon_1^g = 0.$$

Thus, I start the monetary policy impulse-response function with

$$\varepsilon_1 = \left[\begin{array}{cccccc} b_{r^n,i} & b_{g,i}=0 & b_{\pi,i} & \dots & b_{\iota,i}=1 & \dots \end{array}\right]',$$

For the fiscal shock I run the same regression without g_t,

$$\varepsilon^z_{t+1} = b_{z,i}\varepsilon^i_{t+1} + b_{z,pv}a'_s(I-A)^{-1}\varepsilon_{t+1} + \eta_{t+1},$$

And then I start the fiscal impulse-response function with

$$\varepsilon_1 = \left[\begin{array}{cccc} b_{r^n,pv} & b_{g,pv} & b_{\pi,pv} & \dots \end{array}\right]'.$$

Figure 6 presents the responses to the monetary policy shock. Table 3 collects relevant contributions to the inflation identity (29). The instantaneous response of the nominal interest rate $\Delta E_1 i_1$ is 1% by construction, as is the zero instantaneous growth response $\Delta E_1 g_1 = 0$ and the zero response of the sum of surpluses $\Delta E_1 \sum_{j=0}^{\infty} s_{1+j} = 0$. Consumption g then declines, as one might expect from a positive monetary policy shock, and as a new-Keynesian Phillips curve predicts with declining inflation. Although the sum of surpluses does not change by construction, near-term surpluses increase, and long-term surpluses decrease, roughly paralleling the path of consumption. The surplus is procyclical.

Figure 6 shows that

- *The response of inflation to this monetary policy shock is super-Fisherian, with inflation rising immediately.*

A "Fisherian" response has come to mean that if the central bank raises the interest rate i_t, then inflation rises, for example fulfilling the simple Fisher relation $i_t = r + E_t\pi_{t+1}$. A "super-Fisherian" response is one in which raising the interest rate $\Delta E_1 i_1$ raises inflation $\Delta E_1 \pi_1$ immediately. That is the pattern shown in Figure 6. Here, inflation rises by even more than the nominal rate.

A large discount rate effect overcomes the negative inflation effect in the simple constant-real-rate models in section 2.2. and 2.3. Though inflation initially is larger than the expected bond return, starting in year 4, the expected bond return r^n exceeds the interest rate π, and stays there. A negative growth rate g adds to the effective discount rate $r - g$. The sum of the future $r - g$ terms, 2.7%, lowers the value of government debt, producing inflation.

The long-term debt disinflationary effect remains, with instantaneous bond return $r^n_1 - g_1 = -0.99\%$. In the return decomposition on the bottom of Table 3, this return comes from a strong 1.95% future inflation effect, as we would expect from the large, and positive inflation response. If discount rates were constant, this pattern of surpluses and future inflation a negative current inflation $\Delta E_1 \pi_1 = -1.95\%$. That future inflation effect is tempered here with a -0.96% real interest rate effect on bond returns, but still leaves -0.99% disinflationary force. But

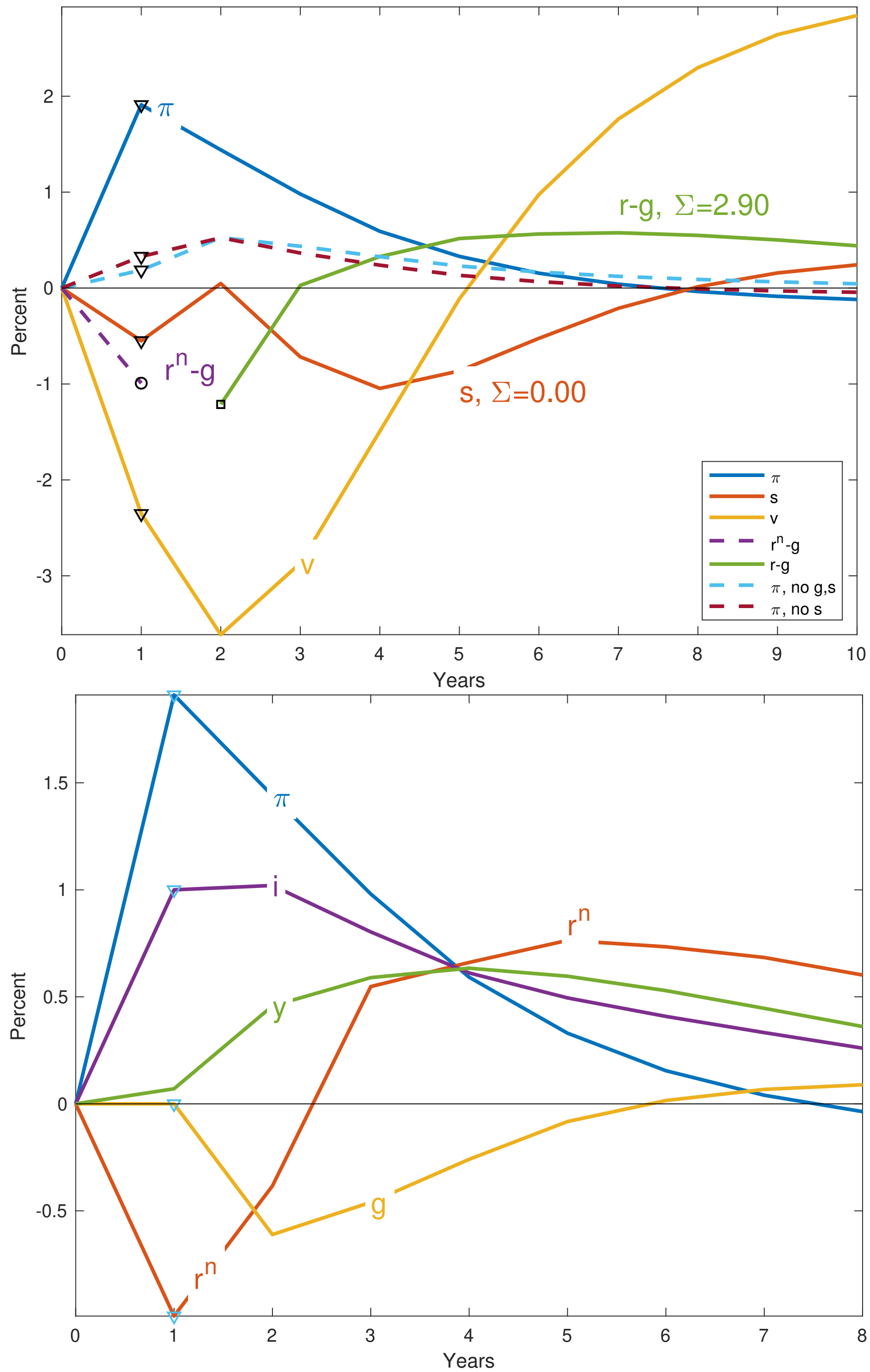


Figure 6: Response to a monetary policy shock – a movement in the interest rate y_1 with no movement in the sum of future surpluses $\sum_1 sv_{1+j}$ or growth g. The dashed lines labeled "no g,s" show the inflation response with movement in growth and surplus.

the large discount rate effect overwhelms this deflationary effect to produce the positive current inflation response.

In sum,

- *A monetary shock, defined as an interest rate rise with no change in surpluses and growth, leads to an immediate and persistent increase in inflation. Unexpected inflation comes from a rise in discount rates, which lowers the value of government debt, overwhelming the negative response that comes from long-term debt and persistent inflation.*

Figure 7 presents responses to fiscal policy shocks. I specify a negative shock to produce positive inflation, which is a clearer story. In the bottom panel, the contemporaneous interest rate response to the fiscal shock is $\Delta E_1 i_1 = 0$, by construction. The interest rate then rises slightly. The fiscal shock gives rise to a positive and persistent inflation.

In the top panel, though the sum of all surplus terms is -1.00% by construction, near-term surpluses rise, and long-term surpluses fall even more. There are three years of negative discount rate movement, offsetting more than half of the surplus shock. This discount rate movement comes from the dynamics shown in the bottom panel. The long-term rate y declines for one period, which gives rise to a large one-period expected return r^n. The remaining -0.50% shock to the present value of surplus results in 0.45% inflation, and a small -0.04% bond return term.

- *A fiscal shock sets off a protracted inflation. Discount rate variation offsets about half of the fiscal shock.*

3.6. Orthogonalization and shock definitions

Most VAR estimates of the effects of a monetary policy shock find small inflation responses. These are often zero or positive in the short run. When inflation responds negatively, it typically drifts down only quite slowly, and even then after a long specification search (Ramey (2016)).

The main difference in these results is that I define a monetary policy shock differently. Standard estimates do not measure fiscal variables or try to keep any measure of fiscal policy constant. In historical episodes, both monetary and fiscal authorities react to the same events. On their own, VARs will thus find monetary policy shocks that also move fiscal variables. This is not a mistake. If one takes a strong passive-fiscal view, then the fiscal authority does not act independently, but always follows the central bank's equilibrium-selection desires, and the central bank can count on it to keep doing so. But if one doubts this mechanism, if one wants to ask what

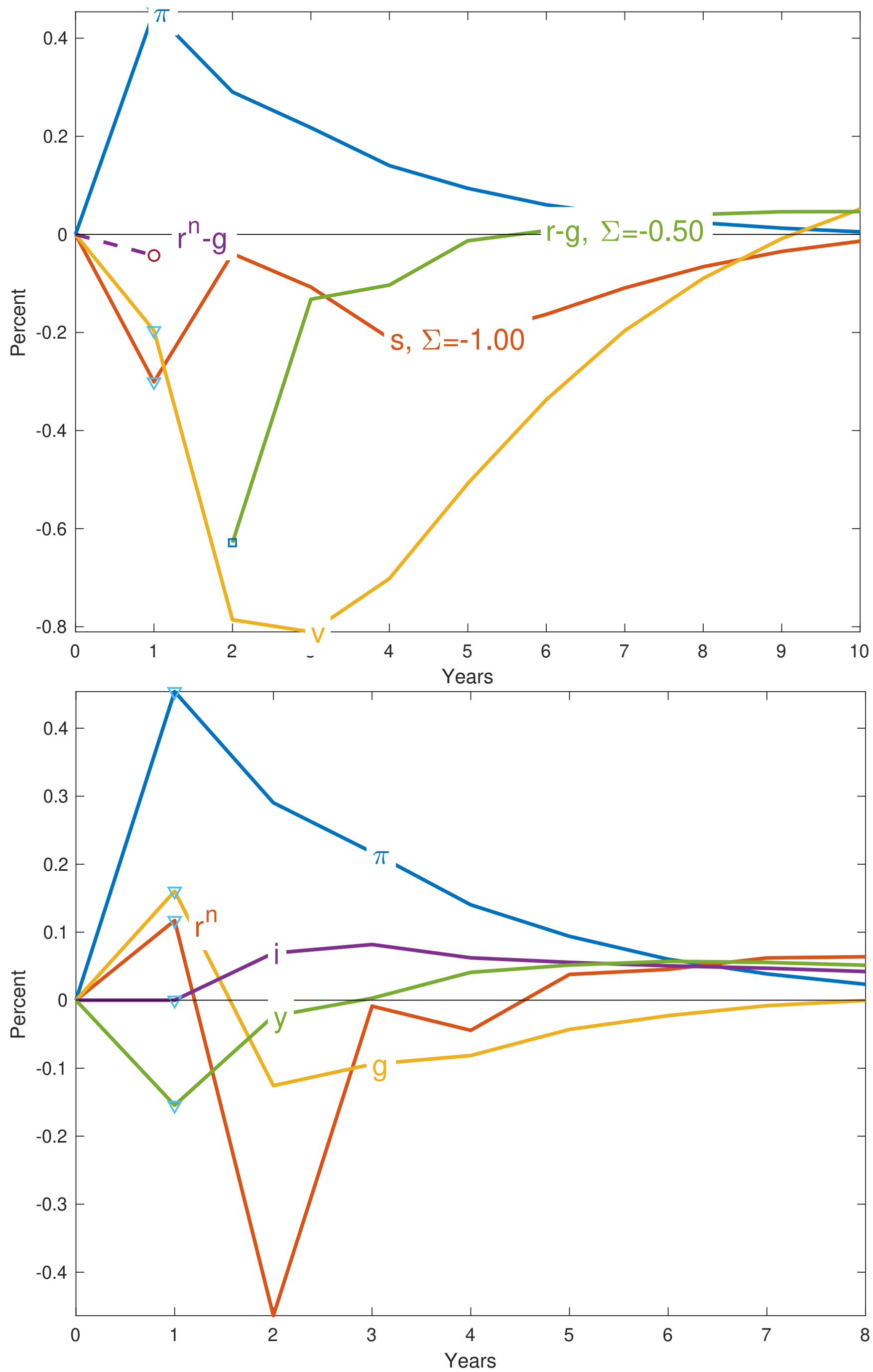


Figure 7: Fiscal policy response. Response to a shock to expected surpluses $\Delta E_{t+1}\sum_{j=1}^{\infty} sv_{t+j} = 1$, with no interest rate shock $v_{t+1}^{y} = 0$.

would happen if monetary policy moved and fiscal policy did not follow, then an orthogonalization such as this one in which monetary policy does not coincide with a fiscal policy change is more interesting. The main innovation of this calculation, then, is try to measure the effects of conceptually separate monetary policy and fiscal shocks, each holding the other constant.

To evaluate this central difference, the dashed lines in the top panel of Figure 6 present the inflation response without restriction on surpluses. The line labeled "no s, no g" also removes the growth orthogonalization, allowing growth to move contemporaneously to the monetary policy shock. These inflation responses with a free fiscal response are much smaller. Surpluses in these cases (not shown) rise sharply, with a sum $\Delta E_1 \sum s_{t+j} = 3.20\%$ and 3.15% respectively. Defining the monetary policy shock to hold surpluses constant is a big part of the difference between these and conventional results.

My calculation is, however, unsophisticated regarding exogeneity and orthogonalization. The contemporaneous correlation between interest rate shocks and shocks to inflation, GDP, or other variables can result from the Fed reacting within the period to those variables, as described for example by a Taylor rule such as (18)

$$i_t = \theta_{i\pi}\pi_t + \theta_{ix}x_t + u_t^i \tag{32}$$

instead of the contemporaneous reaction of the economic variables to the Fed's shock, an innovation to u_t^i. And perhaps even the apparent disturbance u_t^i is taken in response to news about future inflation or output, or other variables such as financial conditions that forecast inflation or output, news not captured by VAR variables.

I define the monetary policy shock so that growth does not respond contemporaneously, conservatively assigning all the correlation between the interest rate and growth to reverse causality from consumption to the interest rate. The correlation is positive, so otherwise we estimate that the interest rate rise raises growth. I can't orthogonalize inflation the same way, or there is by definition no inflation shock left $\Delta E_1 \pi_1 = 0,$ and one of the main questions of the analysis is ruled out. A second reason many VARs find no immediate and small short-run responses of inflation to a monetary policy shock is that they orthogonalize the contemporaneous correlation in this way.

Reality of course lies somewhere in between. Some of the correlation of inflation and interest rate shocks within a year reflects the reaction of interest rates to inflation, so my Fisherian estimate is surely overstated. Some of the correlation of growth and interest rate shocks within a year surely reflects the response of growth to interest rates, not entirely the other way around.

A better estimate thus requires something more sophisticated than a recursive identification, i.e. assuming either that the Fed does not react within a year to specific economic variables, or that those variables do not react within a year to monetary policy shocks. High frequency data, narrative approaches, or a detailed specification and estimation of the policy rule such as (32) may help. However, Ramey (2016) shows that a half-century of effort has still not led to a clear standard answer or procedure, on to which one can layer a fiscal policy assumption. The deepest problem is that the Fed never explains any action as a random innovation. Rather, the Fed always describes every action (or inaction) as a response to something. The only hope is to find a Fed response to something orthogonal to inflation, output, or employment, or forecasts of those, but given those are the Fed's mandate it's hard to think what that object could be.

My shock definitions are even crude relative to the simple theory outlined in section 2.3. There I wrote a monetary policy rule (18), (32) and a fiscal policy rule (19)-(20),

$$s_t = \theta_{s\pi}\pi_t + \theta_{sx}x_t + \alpha v_t^* + u_t^s \tag{33}$$

$$\left[1 + \left(\gamma^{-1} - 1\right)\alpha\right] v_{t+1}^* = v_t^* + i_t - E_t\pi_{t+1} - s_{t+1}.$$

Such a rule captures the clear association of fiscal surpluses to output, via the standard effects of proportional income taxation, and both automatic and discretionary fiscal stabilizers. These are fiscal reactions that one might well wish an analysis of the effects of monetary policy to consider.

Thus, ideally, one should define and estimate monetary and fiscal policy rules such as (32) and (33). One should then define a monetary policy shock as one that comes with no fiscal policy *disturbance* $u_t^s = 0$, and one should define a fiscal shock as one that comes with no monetary policy *disturbance* $u_t^s = 0$, as I did in section 2.3., rather than define the monetary shock as having no change in surpluses s_t and the fiscal shock as having no instantaneous change in the interest rate i_t. But doing so is (at least) a paper-length theoretical, data, and econometric challenge, so I leave it as a suggestion for future research.

The definitions here are not wrong, they just answer different and potentially less interesting questions. The response to the monetary policy shock here answers the question, "what if monetary policy changes and fiscal policy does not change," rather than "... and fiscal policy follows its customary reaction to endogenous variables," and likewise for the fiscal shock.

4. Standard errors

I have delayed a discussion of standard errors because there is nothing important to test. Identities are identities. If $x = y + z$ and x moves, y or z must move, and all we can do is to measure which one. In addition, unlike the case in asset pricing, no important economic hypothesis here rests on whether one of surpluses or discount rates do not move. Standard errors only give us a sense of how accurate the measurement is.

To evaluate sampling distributions I run a Monte Carlo. Most of the interesting statistics – variance decompositions, impulse response functions, $(I - A)^{-1}$, etc. – are nonlinear functions of the underlying data, and the near-unit root in value v_t also induces non-normal distributions. For these reasons, I largely characterize the sampling distribution by the interquartile range – the 25% and 75% points of the sampling distribution.

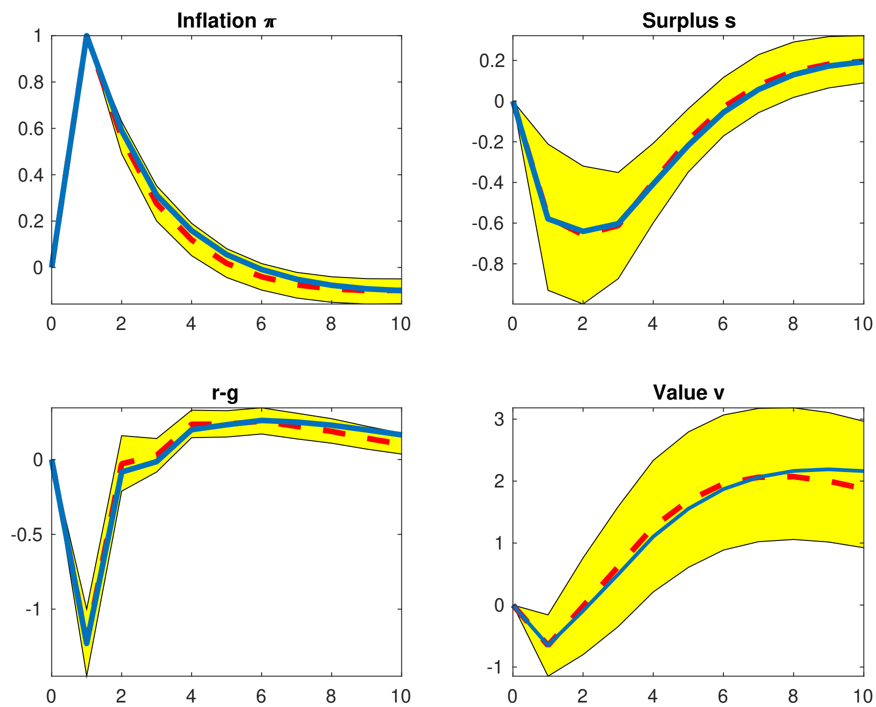


Figure 8: Distribution of the impulse response function, to an inflation shock. The bands are 25% and 75% points of the sampling distribution, the dashed line is the median, and the solid line is the estimate.

Table 4 collects the sampling quantiles for the variance decompositions of Table 3. Figure 8 presents the main components of the impulse-response function relevant to the inflation variance decomposition. The bands are 25% and 75% points of the sampling distribution, the dashed line is the median, and the solid line is the estimate.

As shown in Table 4, the -0.06 sum of future surpluses in the inflation decomposition has quartiles of -0.69 to 0.23. The 1.17 future return contribution has wider quartiles of 0.42 to 1.57. Even the instantaneous return contribution of -0.23 has quartiles of -0.45 to 0.00. That discount rates matter is a pretty solid conclusion, but negative surpluses may contribute more to unexpected inflation than the point estimate suggests.

There are several sources of this rather large sampling variation. First, the shocks are large. As shown in Table 2, the surplus innovation has a 4.08 percentage point standard deviation, and value 6.55 percentage points, compared to 1.12 percentage points for inflation. Our friend $\sigma/\sqrt{T}$ starts off badly.

Second, the shocks are imperfectly correlated. This matters, because in each case I find movements in other variables contemporaneous with the shock of interest by running a regression of the other shocks on the shock of interest. The sampling uncertainty of this orthogonalization adds to that of the VAR. We see a correspondingly wide band around the initial surplus response in Figure 8. There is hope in this observation, however. Higher frequency data can better identify shock correlations, at the cost that one must model the strong seasonal in primary surpluses. Moreover, other shock identifications may have better measured correlations.

Third, we measure sums of future surpluses and discount rates. The value of the debt v_t is the main long-run state variable, and uncertainty about its evolution adds to the uncertainty about the sum of surpluses. The coefficient of value v_t on its own lag is 0.98 in Table 2, so small variations in that value lead to large variation in $(I-A)^{-1}$ sums. The Appendix shows that the last two sources of variation contribute about equally.

Table 4 also presents 25% and 75% quantiles of the inflation decomposition for the recession, monetary, and fiscal shocks of Table 3. The -1.15 surplus response to a recession shock has quantiles -1.28 to 0.49, spanning zero, while the -4.34 discount rate response has quantiles -4.56 to -2.60. The conclusion that discount rate variation is a central part of the story for understanding disinflation during recessions is supported, despite its large sampling error. Similarly, the positive inflation effect and strong discount rate effects of monetary policy shocks are well away from zero, as is the positive inflation effect and discount rate effect of the fiscal shock. The quantiles reveal asymmetric sampling distributions. In many cases the point estimate is well to the edge of the 25%-75% quantiles. The 1.91 inflation effect of monetary policy is even outside the 0.85-1.47 interquartile range.

5. Concluding comments

This analysis evidently just scratches the surface. Quarterly or monthly data are attractive, offering potentially better measurement of correlations and shock orthogonalization but requiring us to model the strong seasonality in surpluses. Debt data go back centuries, allowing and requiring us to think what is the same and different across different periods of history. Inflation through wars and under the gold standard may well have different fiscal foundations than in the postwar environment. A narrative counterpart, especially for big episodes such as the 1970s and 1980s, awaits. Different countries under different monetary and exchange rate regimes and different fiscal constraints will behave differently. A parallel investigation of exchange rates beckons, following Jiang (2019a), Jiang (2019b). One could define shocks in many additional interesting ways. The treatment of debt can be refined in many ways. In particular, the maturity structure is not geometric, and varies over time.

I omitted analysis of the fiscal correlates of the remaining shocks in the VAR. A shock to any other variable, orthogonal to the inflation shock, can move all of the other terms of the inflation identity (3), (29). Such movements must offset: If a shock does not move inflation, but does move the sum of future surpluses, then it must also move the sum of future discount rates or the current bond return, in such a way that current inflation does not move. These additional effects are large. The variation in $\Delta E_1 \sum_{j=0}^{\infty} s_{1+j}$ when other shocks move is large; the corresponding movement in the discount rate term is also large, and the two movements are negatively correlated. The meaning of such orthogonalized movements in expected surpluses, matched by movements in discount rates, in response to these other shocks needs to be understood. I do not pursue this question for length, but also because it is much more interesting if one can give some structural or economic interpretation to the shocks to other variables, which requires a model.

Perhaps most of all, linking these theory-free characterizations to explicit fiscal theory of monetary policy models, or at least to explicit models of discount rates and long-term debt management, is an obviously important step. More broadly, we need to reexamine medium and large scale new-Keynesian DSGE models with fiscal equilibrium selection. This is the natural next step for the fiscal theory of the price level / fiscal theory of monetary project. And, given the immense troubles of the "active money" description of equilibrium selection captured here by (7), perhaps it should be the next natural step for the broader new-Keynesian DSGE model project as well.

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6. Online Appendix to "The Fiscal Roots of Inflation"

6.1. Derivation of the linearized identities

In this appendix I derive the linearized identities (1) (2) and (3),

$$v_{t+1} = v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} - s_{t+1} \tag{34}$$

$$v_t = \sum_{j=1}^{T} s_{t+j} - \sum_{j=1}^{T} \left(r^n_{t+j} - \pi_{t+j} - g_{t+j} \right) + v_{t+T}$$

and

$$\begin{aligned} &\Delta E_{t+1}\pi_{t+1} - \Delta E_{t+1}\left(r^n_{t+1} - g_{t+1}\right) \\ &= -\sum_{j=0}^{\infty} \Delta E_{t+1} s_{t+1+j} + \sum_{j=1}^{\infty} \Delta E_{t+1}\left(r^n_{t+1+j} - \pi_{t+1+j} - g_{t+1+j}\right). \end{aligned} \tag{35}$$

I also define the variables more carefully.

The symbols are as follows:

$$V_t = M_t + \sum_{j=0}^{\infty} Q_t^{(t+1+j)} B_t^{(t+1+j)}$$

is the nominal end-of-period market value of debt, where M_t is non-interest-bearing money, $B_t^{(t+j)}$ is zero-coupon nominal debt outstanding at the end of period t and due at the beginning of period $t+j$, and $Q_t^{(t+j)}$ is the time t price of that bond, with $Q_t^{(t)} = 1$. Taking logs,

$$v_t \equiv \log\left(\frac{V_t}{Y_t P_t}\right)$$

is log market value of the debt divided by GDP, where P_t is the price level and Y_t is real GDP or another stationarity-inducing divisor such as consumption, potential GDP, population, etc. I use consumption times the average GDP to consumption ratio in the empirical work, but I will call Y and ratios to Y "GDP" for brevity.

$$R^n_{t+1} \equiv \frac{M_t + \sum_{j=1}^{\infty} Q_{t+1}^{(t+j)} B_t^{(t+j)}}{M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}} \tag{36}$$

is the nominal return on the portfolio of government debt, i.e. how the change in prices overnight from the end of t to the beginning of $t+1$ affects the value of debt held overnight, and

$$r_{t+1}^{n} \equiv \log(R_{t+1}^{n})$$

is the log nominal return on that portfolio.

$$\pi_t \equiv \log\left(\frac{P_t}{P_{t-1}}\right),\ g_t \equiv \log\left(\frac{Y_t}{Y_{t-1}}\right)$$

are log inflation and GDP growth rate.

We can accommodate explicit default, and yes, the formulas also apply to countries in the Euro. An explicit default is a reduction in the nominal quantity of debt overnight. The $B_t^{(t+j)}$ in the numerator of (36) represents the post-default number of bonds outstanding, i.e. on the morning of time $t+1$, while the $B_t^{(t+j)}$ in the denominator represents the pre-default number of bonds outstanding, i.e. in the evening of time t. A partial default then shows up as a low return. To handle default one would, of course, add notation distinguishing these two values.

Now, I establish the nonlinear flow and present value identities. In period t, we have

$$\sum_{j=0}^{\infty} Q_t^{(t+j)} B_{t-1}^{(t+j)} + M_{t-1} = P_t sp_t + \sum_{j=0}^{\infty} Q_t^{(t+1+j)} B_t^{(t+1+j)} + M_t, \tag{37}$$

where sp_t denotes the real primary (not including interest payments) surplus or deficit. Money M_t at the end of period t is equal to money brought in from the previous period M_{t-1} plus the effects of bond sales or purchases at price $Q_t^{(t+j)}$, less money soaked up by primary surpluses.

The left hand side of (37) is the beginning-of-period market value of debt, i.e. before debt sales or repurchases $B_t^{(t+j)} - B_{t-1}^{(t+j)}$ have taken place. It turns out to be more convenient here to express equations in terms of the end-of-period market value of debt. To that end, shift the time index forward one period and rearrange to write

$$\sum_{j=0}^{\infty} Q_{t+1}^{(t+1+j)} B_t^{(t+1+j)} + M_t = P_{t+1} sp_{t+1} + \sum_{j=0}^{\infty} Q_{t+1}^{(t+2+j)} B_{t+1}^{(t+2+j)} + M_{t+1},$$

$$\sum_{j=1}^{\infty} Q_{t+1}^{(t+j)} B_t^{(t+j)} + M_t = P_{t+1} sp_{t+1} + \sum_{j=1}^{\infty} Q_{t+1}^{(t+1+j)} B_{t+1}^{(t+1+j)} + M_{t+1},$$

$$\left(M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}\right) R_{t+1}^{n} = P_{t+1} sp_{t+1} + \left(M_{t+1} + \sum_{j=1}^{\infty} Q_{t+1}^{(t+1+j)} B_{t+1}^{(t+1+j)}\right),$$

$$\frac{M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}}{P_t Y_t} \frac{R_{t+1}^n}{G_{t+1}} \frac{P_t}{P_{t+1}} = \frac{sp_{t+1}}{Y_{t+1}} + \frac{M_{t+1} + \sum_{j=1}^{\infty} Q_{t+1}^{(t+1+j)} B_{t+1}^{(t+1+j)}}{P_{t+1} Y_{t+1}}. \tag{38}$$

We can iterate this flow identity (38) forward to express the nonlinear government debt valuation identity as

$$\frac{M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}}{P_t Y_t} = \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} \frac{1}{R_{t+k}/G_{t+k}} \right) \frac{sp_{t+j}}{Y_{t+j}}. \tag{39}$$

The market value of government debt at the end of period t, as a fraction of GDP, equals the present value of primary surplus to GDP ratios, discounted at the government debt rate of return less the GDP growth rate. (I assume here that the right hand side converges. Otherwise, keep the limiting debt term or iterate a finite number of periods.)

The nonlinear identities (38) and (39) are cumbersome. I linearize the flow equation (38) and then iterate forward to obtain a linearized version of (39). Write (38) as

$$\frac{V_t}{P_t Y_t} R_{t+1}^n \frac{P_t}{P_{t+1}} \frac{Y_t}{Y_{t+1}} = \frac{sp_{t+1}}{Y_{t+1}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}}.$$

Taking logs,

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log \left(\frac{sp_{t+1}}{Y_{t+1}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}} \right) \tag{40}$$

I linearize in the *level* of the surplus, not its log as one conventionally does in asset pricing, since the surplus is often negative. To linearize in terms of the surplus/GDP ratio, Taylor expand the last term,

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log(e^v + sy) + \frac{e^v}{e^v + sy} (v_{t+1} - v) + \frac{1}{e^v + sy} (sy_{t+1} - sy)$$

where

$$sy_{t+1} \equiv \frac{sp_{t+1}}{Y_{t+1}} \tag{41}$$

denotes the surplus to GDP ratio, and variables without subscripts denote a steady state of (40). With $r \equiv r^n - \pi$,

$$r - g = \log \frac{e^v + sy}{e^v}.$$

Then,

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \left[\log(e^v + sy) - \frac{e^v}{e^v + sy} \left(v + \frac{sy}{e^v} \right) \right] + \frac{e^v}{e^v + sy} v_{t+1} + \frac{e^v}{e^v + sy} \frac{sy_{t+1}}{e^v}$$

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \left[v + r - g - \frac{e^v}{e^v + sy}\left(v + \frac{e^v + sy}{e^v} - 1\right)\right] + \beta v_{t+1} + \beta \frac{sy_{t+1}}{e^v}$$

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = [r - g + (1-\beta)(v-1)] + \beta v_{t+1} + \beta \frac{sy_{t+1}}{e^v} \tag{42}$$

where

$$\beta \equiv e^{-(r-g)}. \tag{43}$$

Suppressing the small constant, and thus interpreting variables as deviations from means, the linearized flow identity is

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \beta \frac{sy_{t+1}}{e^v} + \beta v_{t+1}. \tag{44}$$

Iterating forward, the present value identity is

$$v_t = \sum_{j=1}^{T} \beta^{j-1}\left[\beta \frac{sy_{t+j}}{e^v} - \left(r^n_{t+j} - \pi_{t+j} - g_{t+j}\right)\right] + \beta^T v_T. \tag{45}$$

If we linearize around $r - g = 0$, then the constant in (44) is zero ($sy = 0$), and we obtain the linearized flow and present value identities (34) and (35), with the symbol s_t representing sy_t/e^v. There is nothing wrong with expanding about $r = g$. The point of expansion need not be the sample mean.

To approximate in terms of the surplus to value ratio, write (40) as

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log\left(\frac{V_t}{P_t Y_t}\frac{\frac{sp_{t+1}}{Y_{t+1}}}{\frac{V_t}{P_t Y_t}} + \frac{V_{t+1}}{P_{t+1}Y_{t+1}}\right)$$

$$r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log\left(\frac{\frac{sp_{t+1}}{Y_{t+1}}}{\frac{V_t}{P_t Y_t}} + \frac{\frac{V_{t+1}}{P_{t+1}Y_{t+1}}}{\frac{V_t}{P_t Y_t}}\right)$$

$$r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log\left(sv_{t+1} + e^{v_{t+1} - v_t}\right).$$

At a steady state

$$r - g = \log(1 + sv). \tag{46}$$

$$e^{r-g} = 1 + sv.$$

Taylor expanding around a steady state,

$$r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log(1 + sv) + \frac{1}{(1+sv)}\left(sv_{t+1} - sv + v_{t+1} - v_t\right)$$

$$v_t + (1+sv)\left[r^n_{t+1} - \pi_{t+1} - g_{t+1}\right] = \left[(1+sv)\log(1+sv) - sv\right] + sv_{t+1} + v_{t+1} \tag{47}$$

The linearized flow identity (34) follows, with the symbol s_t representing the surplus to value ratio $s_t = sv_t$, if we suppress the constant, using deviations from means in the analysis, or if we use $r = g$ or $sv = 0$, as a point of expansion.

The linearizations in terms of the surplus to value ratio sv_t are more accurate. The units of the flow identities (34), (44) are rates of return. Dividing the surplus by the previous period's value gives a better approximation to the growth in value, when the value of debt is far from the steady state.

With stationary v_t, the term v_{t+T} does not vanish in (35), where the term $\beta^T v_{t+T}$ vanishes in (45). In this paper, the presence of the v_{t+T} term is not a difficulty. I study innovations $\Delta E_{t+1} v_{t+T}$ and $\Delta E_{t+1} v_{t+T} \to 0$. For other purposes, one may wish to use the surplus to GDP linearization and $r > g$ steady state, so that the limiting term vanishes.

A constant ratio of surplus to market value of debt for any price level path leads to a passive fiscal policy. An unexpected deflation raises the real value of debt. If surpluses always rise in response, they validate the lower price level. Thus, although on the equilibrium path one can describe dynamics via either linearization, if one wants to think about how fiscal-theory equilibria are formed, it is better to describe a surplus that does not react to price level changes, so only one value v_t emerges, as is the case in (45). For such purposes, the surplus to GDP definition is appropriate, as well as adopting a linearization point $r > g$ and $\beta < 1$. It's also better to use the nonlinear versions of the identities for determinacy issues. The analysis of this paper is about what happens in equilibrium, and does not require an active-fiscal assumption, so the difference is irrelevant here.

I infer the surplus from the linearized flow identity (34) so which concept the surplus corresponds to makes no difference to the analysis. The difference is only the accuracy of approximation, how close the surplus recovered from the linearized flow identity corresponds to a surplus recovered from the nonlinear exact identity (40).

6.2. A variance decomposition

I use the elements of the impulse response function and their sums to calculate the terms of the unexpected inflation identity (3). We can interpret this calculation as an decomposition of the variance of unexpected inflation. Multiply both sides of (3) by $\Delta E_{t+1}\pi_{t+1}$ and take expectations, giving

$$var\left(\Delta E_{t+1}\pi_{t+1}\right)-cov\left[\Delta E_{t+1}\pi_{t+1},\Delta E_{t+1}\left(r_{t+1}^{n}-g_{t+1}\right)\right] \tag{48}$$
$$=-\sum_{j=0}^{\infty}cov\left[\Delta E_{t+1}\pi_{t+1},\Delta E_{t+1}s_{t+1+j}\right]+\sum_{j=1}^{\infty}cov\left[\Delta E_{t+1}\pi_{t+1},\Delta E_{t+1}\left(r_{t+1+j}^{n}-\pi_{t+1+j}-g_{t+1+j}\right)\right].$$

Unexpected inflation may only vary to the extent that it covaries with current bond returns, or if it forecasts surpluses or real discount rate.

Dividing by $var\left(\Delta E_{t+1}\pi_{t+1}\right)$, we can express each term as a fraction of the variance of unexpected inflation coming from that term. This decomposition adds up to 100%, within the accuracy of approximation, but it is not an orthogonal decomposition, nor are all the elements necessarily positive. Each term is also a regression coefficient of the other terms on unexpected inflation.

The two approaches give exactly the same result – the terms of (48) are exactly the terms of the impulse-response function, to an inflation shock orthogonalized last, i.e. a shock that moves all variables at time 1 including $\Delta E_1\pi_1$.

To see this fact, write the VAR

$$x_{t+1}=Ax_t+\varepsilon_{t+1} \tag{49}$$

so

$$\Delta E_{t+1}\sum_{j=1}^{\infty}x_{t+j}=(I-A)^{-1}\varepsilon_{t+1}.$$

Let a denote vectors which pull out each variable, i.e.

$$\pi_t=a_\pi' x_t,\ s_t=a_s' x_t, \tag{50}$$

etc. Then the present value identity (3) reads and may be calculated as

$$a_\pi'\varepsilon_{t+1}-(a_{r^n}-a_g)'\varepsilon_{t+1}=-a_s'(I-A)^{-1}\varepsilon_{t+1}+a_{rg}'(I-A)^{-1}A\varepsilon_{t+1} \tag{51}$$

where

$$a_{rg}\equiv a_{r^n}-a_\pi-a_g.$$

We can calculate the variance decomposition (48) by

$$a_\pi'\Sigma a_\pi-(a_{r^n}-a_g)'\Sigma a_\pi=-a_s'(I-A)^{-1}\Sigma a_\pi+a_{rg}'(I-A)^{-1}A\Sigma a_\pi$$

where $\Sigma = cov(\varepsilon_{t+1}, \varepsilon'_{t+1})$, and then divide by $a'_\pi \Sigma a_\pi$ to express the result as a fraction,

$$1-(a_{r^n}-a_g)'\frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi} = -a'_s(I-A)^{-1}\frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi} + a'_{rg}(I-A)^{-1}A\frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi}. \tag{52}$$

To show that this variance decomposition is the same as the elements and sum of elements of the impulse-response function to an inflation shock, orthogonalized last, note that the regression coefficient of any other shock ε^z on the inflation shock is

$$b_{\varepsilon^z,\varepsilon^\pi} = \frac{cov(\varepsilon^z_{t+1}, \varepsilon^\pi_{t+1})}{var(\varepsilon^\pi_{t+1})} = \frac{a'_z \Sigma a_\pi}{a'_\pi \Sigma a_\pi},$$

so the VAR shock, consisting of a unit movement in inflation $\varepsilon^\pi_1 = 1$ and movements $\varepsilon^z_1 = b_{\varepsilon^z,\varepsilon^\pi}$ in each of the other variables is given by

$$\varepsilon_1 = \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi}.$$

We recognize in (52) the responses and sums of responses to this shock. Dividing (48) by the variance of unexpected inflation, or examining the terms of (52), we recognize that each term is also the coefficient in a single regression of each quantity on unexpected inflation.

In an analogous way, we can interpret the responses to other shocks as a decomposition of the *co*variance of unexpected inflation with that shock, based on

$$\begin{aligned} &cov\left(\Delta E_{t+1}\pi_{t+1}\varepsilon_{t+1}\right) - cov\left[\varepsilon_{t+1}, \Delta E_{t+1}\left(r^n_{t+1}-g_{t+1}\right)\right] \\ &= -\sum_{j=0}^{\infty} cov\left[\varepsilon_{t+1}, \Delta E_{t+1} s_{t+1+j}\right] + \sum_{j=1}^{\infty} cov\left[\varepsilon_{t+1}, \Delta E_{t+1}\left(r^n_{t+1+j}-\pi_{t+1+j}-g_{t+1+j}\right)\right]. \end{aligned}$$

This variance decomposition is similar in style to the decomposition of return variance in Campbell and Ammer (1993). To avoid covariance terms, however, it follows the philosophy of the price/dividend variance decomposition in Cochrane (1992), extended to a multivariate context. With $x = y + z$, I explore $var(x) = cov(x, y) + cov(x, z)$ rather than $var(x) = var(y) + var(z) + 2cov(y, z)$.

6.3. Formulas for geometric maturity structure

Here I derive the linearized identity

$$r^n_{t+1} \approx \omega q_{t+1} - q_t,$$

which leads to (4),

$$\Delta E_{t+1} r^n_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1}\left[(r^n_{t+1+j} - \pi_{t+1+j}) + \pi_{t+1+j}\right].$$

I also derive the expectations-hypothesis bond-pricing equations of the sticky-price model, (22) and (23),

$$E_t r^n_{t+1} = i_t$$
$$\omega E_t q_{t+1} - q_t = i_t.$$

Suppose the face value of debt follows a geometric pattern, $B_t^{(t+j)} = B_t \omega^{j-1}$. Then the nominal market value of debt is

$$\sum_{j=1}^{\infty} B_t^{(t+j)} Q_t^{(t+j)} = B_t \sum_{j=1}^{\infty} \omega^{j-1} Q_t^{(t+j)}.$$

Define the price of the government debt portfolio as

$$Q_t = \sum_{j=1}^{\infty} \omega^{j-1} Q_t^{(t+j)}.$$

The return on the government debt portfolio is then

$$R^n_{t+1} = \frac{\sum_{j=1}^{\infty} B_t^{(t+j)} Q_{t+1}^{(t+j)}}{\sum_{j=1}^{\infty} B_t^{(t+j)} Q_t^{(t+j)}} = \frac{\sum_{j=1}^{\infty} \omega^{j-1} Q_{t+1}^{(t+j)}}{\sum_{j=1}^{\infty} \omega^{j-1} Q_t^{(t+j)}} = \frac{1 + \omega \sum_{j=1}^{\infty} \omega^{j-1} Q_{t+1}^{(t+1+j)}}{\sum_{j=1}^{\infty} \omega^{j-1} Q_t^{(t+j)}} = \frac{1+\omega Q_{t+1}}{Q_t}.$$

I loglinearize as

$$r^n_{t+1} = \log\left(\frac{1+\omega Q_{t+1}}{Q_t}\right) = \log\left(1+\omega e^{q_{t+1}}\right) - q_t \approx \log\left(\frac{1+\omega Q}{Q}\right) + \frac{\omega Q}{1+\omega Q}\tilde{q}_{t+1} - \tilde{q}_t \tag{53}$$

where as usual variables without subscripts are steady state values and tildes are deviations from steady state.

In a steady state,

$$Q^{(t+j)} = \frac{1}{(1+i)^j}$$
$$Q = \sum_{j=1}^{\infty} \omega^{j-1} \frac{1}{(1+i)^j} = \left(\frac{1}{1+i}\right)\left(\frac{1}{1-\frac{\omega}{1+i}}\right) = \frac{1}{1+i-\omega}. \tag{54}$$

The limits are $\omega = 0$ for one period bonds, which gives $Q = 1/(1+i)$, and $\omega = 1$ for perpetuities, which gives $Q = 1/i$. The terms of the approximation (53) are then

$$\frac{1+\omega Q}{Q} = 1+i$$
$$\frac{\omega Q}{1+\omega Q} = \frac{\omega}{1+i}$$

so we can write (53) as

$$r^n_{t+1} \approx i + \frac{\omega}{1+i}\tilde{q}_{t+1} - \tilde{q}_t.$$

since $i < 0.05$ and $\omega \approx 0.7$, I further approximate to

$$r^n_{t+1} \approx i + \omega\tilde{q}_{t+1} - \tilde{q}_t. \tag{55}$$

To derive (4), iterate (55) forward to express the bond price in terms of future returns,

$$\tilde{q}_t = -\sum_{j=1}^{\infty} \omega^j \tilde{r}^n_{t+j}$$

Take innovations, move the first term to the left hand side, and divide by ω,

$$\Delta E_{t+1}\tilde{r}^r_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1}\tilde{r}^n_{t+1+j}$$

then add and subtract inflation to get (4),

$$\Delta E_{t+1}\tilde{r}^n_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1}\left[(\tilde{r}^n_{t+1+j} - \tilde{\pi}_{t+1+j}) + \tilde{\pi}_{t+1+j}\right].$$

The expectations hypothesis states that expected returns on bonds of all maturities are the same,

$$E_t r^n_{t+1} = i_t$$
$$i + \omega E_t\tilde{q}_{t+1} - \tilde{q}_t = i_t$$
$$\omega E_t\tilde{q}_{t+1} - \tilde{q}_t = \tilde{\imath}_t$$

The first and third are equations (22) and (23) of the text. In the text, all variables are deviations from steady state, so I drop the tilde notation.

The yield y_t on the government bond portfolio is the i_t that solves (54) for given Q_t,

$$y_t = \frac{1}{Q_t} + \omega - 1$$

To find the yield as deviation from steady state, given the bond portfolio price as deviation from steady state, write

$$\begin{aligned}
q_t &= \log \frac{1}{1+i-\omega} + \tilde{q}_t \\
y_t &= e^{-\log \frac{1}{1+i-\omega} + \tilde{q}_t} + \omega - 1 \\
\tilde{y}_t &= e^{-\log \frac{1}{1+i-\omega} + \tilde{q}_t} - e^{-\log \frac{1}{1+i-\omega}} = \left(e^{\tilde{q}_t} - 1\right)(1+i-\omega).
\end{aligned}$$

6.4. Sticky-price model algebra

Here, I set out the algebra to solve the model (15)-(25). I express the model in the form

$$Ay_{t+1} = By_t + C\varepsilon_{t+1} + D\delta_{t+1} \tag{56}$$

where y is a vector of variables, ε are the structural shocks, and δ are expectational errors in the equations that only tie down expectations. We eigenvalue decompose the transition matrix $A^{-1}B$, we solve unstable roots forward and stable roots backward to determine the expectational errors δ as a function of the structural shocks ε. Then, we can compute the impulse-response function.

First, eliminate redundant variables to write (15)-(25) as

$$x_{t+1} + \sigma\pi_{t+1} = x_t + \sigma\left(\theta_{i\pi}\pi_t + \theta_{ix}x_t + u_t^i\right) + \delta_{t+1}^x + \sigma\delta_{t+1}^\pi$$

$$\pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}x_t + \delta_{t+1}^\pi$$

$$\left[1 + \left(\frac{1}{\gamma} - 1\right)\alpha\right] v_{t+1}^* + \left(\theta_{s\pi}\pi_{t+1} + \theta_{sx}x_{t+1} + \alpha v_{t+1}^* + u_{t+1}^s\right) = \left(\theta_{i\pi}\pi_t + \theta_{ix}x_t + u_t^i\right) - \left(\frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}x_t\right) + v_t^*$$

$$v_{t+1} - \left(\omega q_{t+1} - q_t\right) + \pi_{t+1} + \left(\theta_{s\pi}\pi_{t+1} + \theta_{sx}x_{t+1} + \alpha v_{t+1}^* + u_{t+1}^s\right) = v_t$$

$$\omega q_{t+1} = \left(\theta_{i\pi}\pi_t + \theta_{ix}x_t + u_t^i\right) + q_t + \omega\delta_{t+1}^q,$$

or

$$x_{t+1} + \sigma\pi_{t+1} = \left(1 + \sigma\theta_{ix}\right)x_t + \sigma\theta_{i\pi}\pi_t + \sigma u_t^i + \delta_{t+1}^x + \sigma\delta_{t+1}^\pi$$

$$
\begin{aligned}
\pi_{t+1} &= -\frac{\kappa}{\beta}x_t + \frac{1}{\beta}\pi_t + \delta^{\pi}_{t+1} \\
\theta_{sx}x_{t+1} + \theta_{s\pi}\pi_{t+1} + \left(1+\frac{\alpha}{\gamma}\right)v^*_{t+1} + u^s_{t+1} &= \left(\theta_{ix} + \frac{\kappa}{\beta}\right)x_t + \left(\theta_{i\pi} - \frac{1}{\beta}\right)\pi_t + v^*_t + u^i_t \\
\theta_{sx}x_{t+1} + (1+\theta_{s\pi})\pi_{t+1} + \alpha v^*_{t+1} + v_{t+1} - \omega q_{t+1} + u^s_{t+1} &= v_t - q_t \\
\omega q_{t+1} &= \theta_{ix}x_t + \theta_{i\pi}\pi_t + q_t + u^i_t + \omega\delta^q_{t+1}.
\end{aligned}
$$

In matrix notation (56),

$$
\begin{bmatrix}
1 & \sigma & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\theta_{sx} & \theta_{s\pi} & 1+\alpha/\gamma & 0 & 0 & 0 & 1 \\
\theta_{sx} & 1+\theta_{s\pi} & \alpha & 1 & -\omega & 0 & 1 \\
0 & 0 & 0 & 0 & \omega & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_{t+1} \\ \pi_{t+1} \\ v^*_{t+1} \\ v_{t+1} \\ q_{t+1} \\ u^i_{t+1} \\ u^s_{t+1}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1+\sigma\theta_{ix} & \sigma\theta_{i\pi} & 0 & 0 & 0 & \sigma & 0 \\
-\kappa/\beta & 1/\beta & 0 & 0 & 0 & 0 & 0 \\
\theta_{ix}+\kappa/\beta & \theta_{i\pi}-1/\beta & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
\theta_{ix} & \theta_{i\pi} & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho^i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho^s
\end{bmatrix}
\begin{bmatrix}
x_t \\ \pi_t \\ v^*_t \\ v_t \\ q_t \\ u^i_t \\ u^s_t
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1
\end{bmatrix}
\begin{bmatrix}
\varepsilon^i_{t+1} \\ \varepsilon^s_{t+1}
\end{bmatrix}
+ \begin{bmatrix}
1 & \sigma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & 0 & 0 \\ 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta^x_{t+1} \\ \delta^\pi_{t+1} \\ \delta^q_{t+1}
\end{bmatrix}
$$

Now, we solve the model as

$$\begin{aligned}
Ay_{t+1} &= By_t + \ C\varepsilon_{t+1} + D\delta_{t+1} \\
y_{t+1} &= A^{-1}By_t + A^{-1}C\varepsilon_{t+1} + A^{-1}D\delta_{t+1} \\
y_{t+1} &= Q\Lambda Q^{-1}y_t + A^{-1}C\varepsilon_{t+1} + A^{-1}D\delta_{t+1} \\
Q^{-1}y_{t+1} &= \Lambda Q^{-1}y_t + Q^{-1}A^{-1}C\varepsilon_{t+1} + Q^{-1}A^{-1}D\delta_{t+1} \\
z_{t+1} &= \Lambda z_t + Q^{-1}A^{-1}C\varepsilon_{t+1} + Q^{-1}A^{-1}D\delta_{t+1}
\end{aligned}$$

Let G_f select rows with eigenvalues greater than one, and G_b select rows with eigenvalues less than one. For example, if the first and third eigenvalues are greater than or equal to one,

$$G_f = \begin{bmatrix} 1 & 0 & 0 & ... \\ 0 & 0 & 1 & ... \end{bmatrix}.$$

Then, the z corresponding to eigenvalues greater than one must be zero so

$$\begin{aligned}
0 &= G_f Q^{-1}A^{-1}C\varepsilon_{t+1} + G_f Q^{-1}A^{-1}D\delta_{t+1} \\
\delta_{t+1} &= -\left(G_f Q^{-1}A^{-1}D\right)^{-1} G_f Q^{-1}A^{-1}C\varepsilon_{t+1}
\end{aligned}$$

For this to work there must be as many rows of G_f as columns of δ, i.e. as many eigenvalues greater or equal to one as there are expectational errors. Substituting, we have the evolution of the transformed z variables, i.e. the impulse response function,

$$z_{t+1} = \Lambda z_t + Q^{-1}A^{-1}\left[I - D\left(G_f Q^{-1}A^{-1}D\right)^{-1} G_f Q^{-1}A^{-1}\right] C\varepsilon_{t+1},$$

and then the original variables from

$$y_t = Qz_t.$$

I include two small refinements. First, for computation it is better to force the elements of z_t that should be zero to be exactly zero. Machine zeros $(1e - 14)$ multiplied by explosive eigenvalues eventually explode. Thus, I find the non-zero z only by simulating forward the nonzero elements of z,

$$G_b z_{t+1} = G_b \Lambda z_t + G_b Q^{-1}A^{-1}\left[C - D\left(G_f Q^{-1}A^{-1}D\right)^{-1} G_f Q^{-1}A^{-1}C\right] \varepsilon_{t+1}.$$

Second, the consumer's transversality condition tells us that debt v_t cannot explode. There is no reason to impose that the latent state variable v_t^* cannot explode or have a unit root. In solving the model for some parameter values it is important not to unwittingly impose that condition. The most obvious example occurs for passive fiscal policy, if $s_t = .. + \alpha v_t + ...$, not $s_t = .. + \alpha v_t^* + ...$ Then $v_t^* + s_{t+1} + .. = v_t^*$ has a unit root (or explosive in the usual model with discounting), but the quantity v_t^* enters nowhere else in the model. We seem to get determinacy by adding a useless unit root variable.

Rather than $\lim_{T\to\infty} E_{t+1}y_{t+T} = 0$, we need to impose

$$\lim_{T\to\infty} RE_{t+1}y_{t+T} = 0$$

where R is of the form

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{bmatrix},$$

i.e. omitting the row of y_t corresponding to v_t^* (or any other variable that can explode). Then, rather than simply setting to zero the z corresponding to unit and greater eigenvalues, we need to set only

$$\lim_{T\to\infty} RQE_{t+1}z_{t+T} = \lim_{T\to\infty} RQ\Lambda^T z_{t+1} = 0.$$

Denote by $\lambda_{<1}$ the eigenvalues less than one and $\lambda_{>1}$ the eigenvalues greater than one, and similarly for the corresponding z. We want, for example,

$$\lim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} Q \begin{bmatrix} \lambda_{<1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{<1} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{>1} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{>1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{>1} \end{bmatrix}^T \begin{bmatrix} z_{<1} \\ z_{<1} \\ z_{>1} \\ z_{>1} \\ z_{>1} \end{bmatrix} = 0.$$

(The actual system is larger.)

Let G_f^* denote a matrix with ones in the place of eigenvalues greater or equal to one and zeros elsewhere, for example,

$$G_f^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is the matrix G_f above with zero rows added back. A simple test whether this problem is occurring is whether the rank of RQG_f^* is the same as the rank of QG_f^*, i.e. of G_f^* itself since Q is full rank. If that test succeeds, then we are not using the false condition that v^* may not explode to set a linear combination of the z to zero.

If that test fails, then in place of setting $G_f z_{t+1} = 0$, we set $RQG_f^* z_{t+1} = 0$, i.e.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} Q \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_{<1} \\ z_{<1} \\ z_{>1} \\ z_{>1} \\ z_{>1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Express the matrix on the right hand side in row-echelon form, delete the rows with zeros, and proceed as before.

6.5. Sources of sampling variation

Table 5 includes the regression of other shocks on inflation shock that starts off the main inflation decomposition, and thus determines the instantaneous response in Figures 4 and 8. The table also includes the correlation matrix of the shocks.

To measure the relative contribution of the shock correlation and the long-run response function given the shock identification as sources of variation, Table 6 includes two other sampling calculations. The “no b” columns resample data using the original regression of shocks ε^z_{t+1} on inflation shocks ε^π_{t+1}, the top row of Table 5, in each sample. The VAR coefficients still vary across samples, but the identification of the inflation shock does not. The “no A” columns likewise keep constant the VAR regression coefficients, but reestimate the shock regression in each sample. Turning off either source of sampling variation reduces that variation, but not as much as you might think. Sampling variation is still large in either case, and variances add, not

	r^n	g	π	s	v	i	y
	Regression of other shocks on inflation shock						
Coefficient	-0.56	-0.33	1.00	-0.58	-0.65	0.24	0.23
Std. err.	(0.24)	(0.17)	(0.00)	(0.53)	(0.74)	(0.14)	(0.09)
	Correlation matrix of VAR shocks						
r^n	1.00	-0.25	-0.29	-0.27	0.63	-0.74	-0.93
g	-0.25	1.00	-0.24	0.39	-0.56	0.41	0.20
π	-0.29	-0.24	1.00	-0.14	-0.11	0.21	0.31
s	-0.27	0.39	-0.14	1.00	-0.88	0.35	0.26
v	0.63	-0.56	-0.11	-0.88	1.00	-0.63	-0.60
i	-0.74	0.41	0.21	0.35	-0.63	1.00	0.75
y	-0.93	0.20	0.31	0.26	-0.60	0.75	1.00

Table 5: Regression of other shocks on inflation shock, and correlation matrix of VAR shocks

	Fraction			No b		No A	
Component	Estimate	25%	75%	25%	75%	25%	75%
Inflation π_1	1.00						
Bond return $(r_1^n - g_1)$	-0.23	-0.45	0.00	-0.23	-0.23	-0.45	0.00
Future Σs	-0.06	-0.69	0.23	-0.60	0.14	-0.69	0.23
Future $\Sigma r - g$	1.17	0.42	1.57	0.63	1.37	0.42	1.57

Table 6: Decomposition of unexpected inflation variance – distribution quantiles. No b holds the initial response constant across trials. No A holds the VAR regression coefficients constant across trials

standard deviations. Moreover the sampling variation associated with shock orthogonalization – the "no A" exercise – does not go away no matter how small the shocks. Both left and right hand sides of the shock on shock regressions get smaller at the same rate.