A Fiscal Theory of Monetary Policy with Partially-Repaid Long-Term Debt

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Abstract

I construct a simple model with sticky prices, fiscal theory of the price level, interest rate targets, and long-term debt. The fiscal surplus responds endogenously to inflation and output, and fiscal surpluses rise following periods of deficit to repay accumulated debt, but does not respond to arbitrary unexpected inflation and deflation. The model produces reasonable responses to fiscal and monetary policy shocks, including smooth and protracted disinflation following monetary or fiscal tightening.

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1. Introduction

This paper advances the fiscal theory of monetary policy, combining three ingredients that bring us towards an empirically realistic model useful for policy analysis.

A “fiscal theory of monetary policy” uses the new-Keynesian/DSGE ingredients, including rational expectations and market clearing, price stickiness, and a central bank that follows an interest rate target, but a model that substitutes active fiscal for active monetary policy to select equilibria, following the fiscal theory of the price level.

I develop the model and I analyze of the effects of fiscal and monetary policy shocks. However, I use the most standard IS and Phillips curves, despite their well-known empirical shortcomings, to focus on the novelties on the fiscal side, and to understand clearly how they modify this familiar playground.

The first key ingredient is long-term debt with a geometric maturity structure. Long-term debt allows the model to produce a negative response of inflation to higher interest rates, without specifying a negative fiscal shock contemporaneous to the monetary policy shock as the conventional new-Keynesian model does. With long-term debt, higher nominal interest rates lower the nominal value of debt. If there is no change to surpluses or real interest rates, the real value of debt is unchanged. The price level declines so debt regains its real value. At the original price level, people want to buy more debt, and to do so they demand fewer goods and services.

The geometric-maturity specification bridges the gap between Sims (2011) and Cochrane (2017b), who specify perpetuities, and the bulk of the literature which uses one-period debt. US debt is poorly modeled as a perpetuity, but more poorly modeled as overnight debt. Thus, for a quantitatively reasonable model that can produce a negative response of inflation to interest rates, it is important to model debt that is long-term but not too long term.

Long-term debt is especially important to apply the model at relatively high frequencies. Long-term debt allows a fiscal shock to slowly devalue debt via a persistent inflation rather than a price-level jump.

The second and more novel ingredient is a fiscal policy process in which the government repays part current deficits with future surpluses. Yet fiscal policy remains “active,” in the sense of Leeper (1991): The government debt valuation equation determines unexpected inflation, and provides the extra forward-looking root needed to uniquely determine equilibria.

One way to view the fiscal policy specification is that the surplus responds to a latent state variable, which is what the value of debt would be if it accumulated deficits at the expected government bond return, ignoring variation in the real value of debt due to arbitrary unexpected
If surpluses respond to the value of debt, then debts are repaid, and the real value of nominal government debt equals the present value of future surpluses for any initial price level. Fiscal policy is passive. That assumption seems sensible, at least for normal times: Don’t governments with larger and larger debts eventually wake up and raise surpluses to pay them off, at least on average and in expectation?

But that familiar assumption is a sufficient condition for passive policy, not a necessary condition. It hides an implicit assumption, that surpluses respond equally to all changes in the value of the debt. It specifies that governments respond equally to debts accumulated from previous deficits and to changes in the real value of debt induced by unplanned, unexpected, undesired, or multiple-equilibrium inflation and deflation. Active fiscal policy requires only that governments refuse to respond to the latter. By breaking that assumption, by allowing governments to respond to increased debts resulting from past deficits or variation in real interest rates, but not to respond to changes in the real value of debt induced by arbitrary unexpected inflation and deflation, I preserve active fiscal policy and allow governments to commit to repay today’s debts with tomorrow’s surpluses, in full or in part.

Once you consider it, this separation is reasonable. Yes, governments may and often do raise surpluses after a time of deficits. Governments often raise revenue from debt sales, and the value of debt increases after such sales, which essentially proves that investors believe surpluses will rise to pay off new debts. Raising surpluses after such debts have been incurred is only making good on the explicit or implicit promise made when borrowing, and sustains the reputation needed for future borrowing. And we see many institutions in place to try to guarantee or pre-commit to repayment, rather than default or inflation.

But the same government may well refuse to accommodate arbitrary unexpected inflation and deflation. Pre-committing not to raise taxes or cut spending to pay a deflation-induced windfall to nominal bondholders or pre-committing not to cash in on unexpected inflation-induced debt devaluation, allows the government to produce a stable price level.

And we can see institutions and reputations at work to make this commitment as well. A gold standard is a commitment to raise taxes to buy gold in the event of inflation. An inflation target works similarly today. An inflation target signal’s the treasury and government’s commitment to raise taxes to pay off debts at the inflation target, but not to raise taxes in the event of deflation. It is not just a central bank commitment device. One hears calls now for governments to commit to helicopter-drop fiscal stimulus in event of deflation. A fiscal rule that raises taxes in response to inflation, and runs such stimulus in the event of deflation, but still pays off inflation.
debts incurred from past deficits should the price level come out on target implements the sort of response I have in mind. And even without formal commitments, such a refusal is a sensible set of expectations. Should a 50% cumulative deflation break out, likely in a severe recession, does anyone expect the government to sharply raise taxes or drastically cut spending, to pay an unexpected, and, it will surely be argued, undeserved, windfall to nominal bondholders? (The Eggertsson (2008) account of the 1933 Roosevelt Administration deflation-fighting policy package can be read as just such a refusal.)

A second and equivalent way to view the fiscal policy specification is that fiscal policy shocks have an S-shaped moving average representation, so that today’s deficits lead to expectations of future surpluses. Here we think of surpluses responding to their own past, not to some external variable. Almost all of the fiscal theory literature specifies AR(1) or similar positively autocorrelated process for the disturbance to surpluses. But the AR(1) is a terrible model for fiscal surpluses. It predicts that inflation occurs with deficits, and deflation occurs with surpluses, precisely the opposite pattern we see in data across the business cycle. Worse, it predicts that a higher surplus raises the value of the debt, because a higher surplus forecasts higher future surpluses, and the value of the debt is the present value of future surpluses. Yet higher surpluses in the data unequivocally pay down the value of the debt (Canzoneri, Cumby, and Diba (2001)). With an AR(1) surplus process, debt sales raise no revenue but merely drive down bond prices. That debt sales raise revenue is direct evidence that investors expect today’s deficits to lead to higher surpluses at some point in the future. With an AR(1) surplus process, all deficits are paid for by devaluing outstanding debt via inflation, and none are paid for by selling bonds. Yes, one might overcome these predictions with strong countervailing discount rate effects, but it would be better to start with a model that does not have so clearly the wrong signs of such basic predictions.

Within the standard class of fiscal policies then – surpluses respond more or less to all sources of variation in debt, and follow AR(1) shocks – we are left with a conundrum. Fiscal theory leaves us deeply counterfactual predictions. But any response of surpluses to debt which solves those predictions destroys the fiscal theory by leading to passive fiscal policy.

Well, let us abandon the AR(1) model for surplus disturbances, or, equivalently, let us abandon the assumption that surpluses must respond equally to all sources of variation in the real market value of debt! A S-shaped moving average allows for a sensible model in which governments raise revenue to fund today’s deficits by selling bonds, which they pay off with larger future surpluses. Yes, such processes are much harder to estimate, but all the more reason we should think hard about them and suspect contrary estimates.
But we do not want to specify that all deficits are eventually repaid by future surpluses, or there are no fiscal shocks and, absent discount rate variation, no unexpected inflation. I write down a surplus process that is S-shaped, but in which the rise in future surpluses is not quite as large as the current deficits. In this way, part of a fiscal shock is financed by inflating away outstanding bonds, and part is financed by promising higher future surpluses.

The third key and somewhat novel ingredient is discount rate variation. The fiscal theory relates unexpected inflation to revisions in the present value of future surpluses. But changes in present values can result from discount rate changes as well as changes in expected future surpluses. When the discount rate for government debt declines, the same surpluses become more valuable, a deflationary force. People want to buy government bonds. To do so, they demand fewer goods and services, lowering the price level. Cochrane (2019) finds that most time-series variation in unexpected US inflation comes from discount rates, not changes in expected surpluses. When inflation falls in a recession, for example, with large deficits, real interest rates decline. This decline raises the value of government debt and accounts for all of the persistent deflation in a recession.

Discount rate variation is also important in this paper, to the model's predictions of the effects of fiscal and monetary policy under sticky prices. For example, a rise in nominal interest rates with sticky prices raises real interest rates. That rise raises the discount rate for government debt, an inflationary force.

The main results of this paper are responses to persistent fiscal and monetary policy shocks. A negative shock to fiscal surpluses leads to a protracted inflation, and via the Phillips curve an output expansion. When monetary policy reacts to inflation, it moderates that inflation, but spreads the inflation further forward. Today's deficits do lead to a long string of future surpluses, and the surplus seems to respond to debt. Both observations could lead one to falsely infer a passive fiscal regime.

A monetary policy shock – a persistent rise in interest rates that does not shock fiscal surpluses – leads to a protracted disinflation, and an output decline. Endogenous surplus variation amplifies the deflationary impact of monetary policy. Endogenous discount rate variation buffers it. The rise in nominal rates raises real rates which discount surpluses more highly, an inflationary force.

Since the IS and Phillips curves are so stylized, and shrinking from the formidable identification and estimation difficulties of all such models, I do not proceed to estimation and testing, or of more rigorously comparing impulse responses to even my own estimates (Cochrane (2019)). But the fact that a fiscal theory of monetary policy model can be so easily built, can sur-
mount classic criticisms, and produces reasonable responses, avoiding pathological predictions such as price level jumps and surpluses that raise rather than lower the value of debt, is already important news.

In this way, this paper's point is also methodological. You may have recognized problems with standard new-Keynesian model. (My own list features Cochrane (2011), Cochrane (2017a), Cochrane (2018), but there are hundreds more.) You may have found active fiscal rather than active money an appealing route to avoid these problems. But you may have been daunted by theoretical controversies that pervade the literature, the impression that the model would make immediately silly predictions, or the impression that you would have to do something fundamentally hard and different. This paper shows by example that constructing realistic fiscal theory of monetary policy models is a nearly trivial modification of the parallel construction of new-Keynesian DSGE models. The questions one is led to ask, and the answers, are potentially quite different.

1.1 Literature

The fiscal theory of monetary policy, uniting fiscal theory with interest rate targets and sticky prices, is a recent development in a long literature on the fiscal theory of the price level.

What's new in this paper? The surplus process is the central novel ingredient. The analysis of responses to monetary and fiscal policy in this paper, tracking the effects of discount rate variation and long-term debt, is also novel.

Canzoneri, Cumby, and Diba (2001) forcefully point out the pathologies induced by an AR(1) surplus process – that higher surpluses are predicted to raise, not lower, the value of debt. They acknowledge that an S-shaped surplus process can fix the problem, but do not estimate such a process or put it in a model. Cochrane (2001) notes the need for an S-shaped surplus process, and that such a process can result in a non-invertible moving average representation. But that paper does not implement the S-shaped surplus process. It also does not describe an economy with interest rate targets or sticky prices.

As long as one has an AR(1) or other positively correlated surplus process, the Canzoneri, Cumby, and Diba (2001) conundrum applies. The fiscal regime will have counterfactual implications. Those counterfactual implications will contaminate estimates and tests. (At a general level, fiscal and monetary regimes are observationally equivalent, so tests between them or estimates of Markov-switches depend crucially on such supporting assumptions.) Yet if one fixes the counterfactual implications by allowing surpluses to respond to the value of debt, one loses active fiscal policy.
In the fiscal theory of monetary policy, Sims (2011) and Cochrane (2017b) are immediate antecedents, including sticky prices, long-term debt, and computing the response to monetary and fiscal policy shocks. This paper both builds on and simplifies those models. Both of those models specify perpetuities and an exogenous AR(1) surplus, and neither details the separate effects of surplus and discount rate shocks.

A variety of more complex models combining (sometimes time-varying) fiscal price determination and interest rate targets have been built. While these have included many of the features here, none has included them all and especially the S shaped surplus process or the equivalent distinction between reacting to debt and reacting to unexpected inflation. Thus, they all embody the AR(1) surplus conundrum.

Leeper, Traum, and Walker (2017) is a detailed a sticky-price model allowing fiscal theory solution, aimed at evaluating the output effects of fiscal stimulus. But they specify fiscal policy as an AR(1), that can only respond to all values of the debt if it responds at all (p. 2416).

Their paper, and others that include a surplus that responds to output, includes an indirect mechanism that buffers the AR(1) surplus conundrum somewhat: An initial deficit leads to inflation, which raises output, which raises tax revenues, and leads to higher surpluses.

The comprehensive survey in Leeper and Leith (2016) studies the standard IS-Phillips curve model of this paper, including a monetary policy rule with endogenous responses, but again surpluses can only respond to the full value of debt, leading to passive policy, or follow an AR(1).

Bianchi and Melosi (2017) specify that taxes follow an AR(1) that responds to output, as it does here, and switches between a passive-fiscal regime that responds to debt and an active-fiscal regime that does not (Equation (6) p. 1041). Government spending also follows an AR(1) that responds to output (p. 1040). They also use one-period debt. Their paper is centrally about the absence of deflation in response to a preference shock, and how expectations of a switch between regimes affects responses to shocks. They don't analyze monetary policy shocks.

Moreover, this paper like all Markov-switching regimes paper is not obviously fiscal. What exactly goes wrong asymptotically at an off-equilibrium inflation rate, inflation or the real value of government debt? As Davig and Leeper (2006) point out, an economy which appears even to have both policies passive can be determinate, if people expect at some future date to switch to one or the other active regime. Much new-Keynesian literature at the zero bound gains determinacy by imagining a switch to active monetary policy at the end of the zero bound.

In the context of this and similar papers, this paper asks less – before we Markov-mix regimes, and add a zero bound, let's figure out how monetary and fiscal policy work in a pure
fiscal regime.

These remarks are not criticisms. I just need to establish that this apparently simple analysis in this paper is actually somewhat novel.

Much of the fiscal theory literature has pursued various theoretical controversies. A big point of this paper and surrounding literature is to begin productively use fiscal theory to productively understand US data and policy choices.

2. Ingredients

The central model of this paper appears in Section 3. I build to that model with a few simpler models. These models explore the key ingredients and important intuition of the final model in clearer simple settings, they summarize lessons of the literature learned in more complex settings, and they also have deficiencies that motivate the more complex ingredients of the final model.

2.1 Linearized identities

I start with a set of linearized identities to handle long-term debt and time-varying interest rates. These identities are derived in the Appendix to Cochrane (2019).

Start with a linearized version of the government debt flow identity,

\[ \rho v_{t+1} = v_t + r^n_{t+1} - \pi_{t+1} - s_{t+1}. \]  (1)

Log debt at the end of period \( t+1 \), \( v_{t+1} \), is equal to its value at the end of period \( t \), \( v_t \), increased by the log nominal return on the portfolio of government bonds \( r^n_{t+1} \) less inflation \( \pi_{t+1} \), and less the primary surplus\(^1\) \( s_{t+1} \). The parameter \( \rho = e^{-\tau} \) is a constant describing the linearization point.

The main model below uses only (1) as the matrix solution method implicitly solves it forward and imposes stability. However, it is useful to solve it forward explicitly, as this allows us to solve simpler models analytically and to understand the mechanisms behind computed solutions of the larger model.

Iterating (1) forward, we have a present value identity,

\[ v_t = \sum_{j=1}^{\infty} \rho^{j-1} s_{t+j} - \sum_{j=1}^{\infty} \rho^{j-1} \left( r^n_{t+j} - \pi_{t+j} \right). \]  (2)

\(^1\)More precisely, \( s_{t+1} \) is \( \rho \) times the real primary surplus divided steady state debt. For brevity, I refer to \( s_{t+1} \) as simply the "surplus." Cochrane (2019) defines debt and surplus to GDP ratios, and the return \( r^n_t \) becomes \( r^n_t - g_t \). For simplicity I abstract from growth here, but it is an important transformation for empirical work.
The log value of government debt is the present value of future surpluses, discounted at the ex-
post real return.

Taking time $t+1$ innovations $\Delta E_{t+1} \equiv E_{t+1} - E_t$ and rearranging, we have an unexpected
inflation identity,

$$\Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} r^n_{t+1} = - \sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} s_{t+1+j} + \sum_{j=1}^{\infty} \rho^j \Delta E_{t+1} \left( r^n_{t+1+j} - \pi_{t+1+j} \right).$$

(3)

A decline in the present value of surpluses on the right hand side must correspond to a lower real
value of the debt on the left hand side. This reduction can come about by unexpected inflation, or by a decline in nominal long-term bond prices and therefore a low return $r_{t+1}$.

The second term on the right hand side captures discount rate effects. If expected real
returns on government bonds rise, the present value of surpluses declines. Government bonds
become less valuable. People try to sell bonds, leading either to lower bond prices, a low $r_{t+1}$, or
to higher inflation $\pi_{t+1}$. Cochrane (2019) finds that such discount rate variation is the dominant
source of movement in US time series data on inflation.

When does a decline in the present value of surpluses result in inflation, vs. a decline in
nominal bond prices? To understand this question, I use a second linearized identity. With a
geometric maturity structure of the debt, in which the face value of maturity $j$ debt declines at
rate $\omega^j$, we have

$$\Delta E_{t+1} r^n_{t+1} = - \sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} r^n_{t+1+j} = - \sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \left( r^n_{t+1+j} - \pi_{t+1+j} \right) + \pi_{t+1+j}.$$  

(4)

An unexpectedly low bond return $\Delta E_{t+1} r^n_{t+1}$ corresponds mechanically to higher expected fu-
ture nominal returns, as bond prices are the inverse of bond yields. And nominal expected re-
turns equal real returns plus inflation. Cochrane (2019) finds that variation in expected inflation
is usually the dominant term in this decomposition. Fiscal and monetary shocks can set off a
protracted inflation, which slowly devalue or revalue long term bonds, with nominal bond prices
essentially marking the future inflation to market in (3). This scenario is much more realistic
than one-period price-level jumps or iid inflation. Real expected return variation can matter as
well, however. We will see similar effects in the theoretical models of this paper.

We can eliminate the bond return from (3)-(4) to yield

$$\sum_{j=0}^{\infty} \omega^j \Delta E_{t+1} \pi_{t+1+j} = - \sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} s_{t+1+j} + \sum_{j=1}^{\infty} \left( \rho^j - \omega^j \right) \Delta E_{t+1} \left( r^n_{t+1+j} - \pi_{t+1+j} \right).$$  

(5)
This equation emphasizes how, in the presence of long-term debt $\omega > 0$ a fiscal shock can be met by drawn out inflation rather than a price level shock. It also shows how absent surplus news, the government may still be able to substitute future for current inflation.

These equations are all identities, and they hold for passive fiscal as well as active fiscal models. One can interpret them as describing the potential mechanisms of passive fiscal adjustment. Since (1) and (2) hold ex-post, these equations also hold with respect to any information set generating expectations $E$.

### 2.2 Simplest FTMP model and active vs. passive policies

Start with flexible prices, a constant real interest rate and expected bond return $E_t r_{t+1}^n = E_t \pi_{t+1}$, and one-period debt $\omega = 0$. The model is composed of only the Fisher equation, (the linearized intertemporal first-order condition)

$$i_t = E_t \pi_{t+1},$$  \hspace{1cm} (6)

and the inflation identity (3), which simplifies to:

$$\Delta E_{t+1} \pi_{t+1} = -\Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j s_{t+1+j}. $$  \hspace{1cm} (7)

(With one period debt, $i_t = r_{t+1}^n$ so $\Delta E_{t+1} r_{t+1}^n = 0$. These are deviations from steady state, so a term $r$ on the right hand side of (6) is zero.)

If the central bank sets an interest rate target $\{i_t\}$, it determines expected inflation by (6). If fiscal policy is passive, meaning that surpluses adjust ex-post so that (4) holds for any unexpected inflation $\Delta E_{t+1} \pi_{t+1}$ then this model leaves multiple equilibria, as any value of unexpected inflation $\Delta E_{t+1} \pi_{t+1}$ can occur.

The standard new-Keynesian approach solves this multiplicity by specifying an active monetary policy

$$(i_t - i_t^*) = \phi (\pi_t - \pi_t^*), \phi > 1, $$  \hspace{1cm} (8)

where $\pi_t^*, i_t^*$ are equilibrium values (i.e. satisfying $i_t^* = E_t \pi_t^*)$ that the central bank wishes to select. One also adds a rule against nominal explosions, that $\lim_{T \to \infty} E_{t+1} \pi_{t+T}$ must be finite. Now only one value of unexpected inflation remains (Woodford (2003), Cochrane (2011)).

For example, suppose the central bank wishes to produce an AR(1) inflation process,

$$\pi_{t+1}^* = \theta \pi_t^* + \varepsilon_{t+1}. $$  \hspace{1cm} (9)
By (6), the equilibrium interest rate must follow
\[ i_t^* = \theta \pi_t^*. \tag{10} \]

The central bank cannot simply set a time-varying peg following (10), however, as this specification would not determine unexpected inflation \[ \Delta E_{t+1} \pi_{t+1} = \varepsilon_{t+1}. \]

The central bank also specifies \( \phi > 1 \) and announces (8), that should another inflation \( \pi_{t+1} \neq \pi_{t+1}^* \) emerge, the central bank will lead the economy to hyperinflation or deflation. The latter provision and the rule against nominally explosive equilibria selects (9) as the unique equilibrium.

Equation (8) is more commonly written
\[ i_t = \phi \pi_t + v_t, \tag{11} \]
with \( v_t \equiv (\theta - \phi) \pi^*. \) I write it in the equivalent form (8) to emphasize that “monetary policy,” the interest rate rule (10) that we observe in equilibrium, is separate from “equilibrium selection policy,” the threat (8), unobserved in equilibrium, that the central bank uses to select one of multiple equilibria. This expression motivates my parallel writing of fiscal policy that responds to changes in real value of debt brought about by deficits and desired inflation, and does not respond to off-equilibrium inflation.

A fiscal theory of monetary policy specifies instead a passive \( \phi < 1 \) equilibrium-selection policy (8) – or, really no such policy at all, \( \phi = 0 \), just erasing (8) – and turns off the passive fiscal assumption, so (7) does not hold automatically. Now the combination (6) and (7) uniquely determine both expected and unexpected inflation.

Central banks remain powerful in this model. Central banks cannot directly affect fiscal policy, and central banks can no longer count on a passive fiscal adjustment to validate an attempt at equilibrium-selection policy. But central banks still set interest rate targets\(^2\), and thereby they control expected inflation via (6). Fiscal events only determine unexpected inflation, the instantaneous response of inflation to a shock \( \Delta E_{t+1} \pi_{t+1} \).

The fiscal and monetary policies in this model can easily be generalized. Interest rate tar-

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\(^2\)How does the central bank set an interest rate target? Even in this cashless and frictionless model, the central bank can set interest rates by varying the quantity of debt, without changing surpluses. Briefly, for example, writing the nonlinear valuation identity with a constant interest rate as
\[ \frac{B_{t-1}}{P_t} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}, \]
then a change in \( B_{t-1} \) at time \( t - 1 \) with no change in surpluses changes expected inflation \( E_{t-1}(P_{t-1}/P_t) \) and therefore the nominal interest rate. See Cochrane (2017b) Section 2.4 for an extended discussion. Alternatively, one may appeal to the Woodford (2003) cashless limit.
gets may also follow rules and react endogenously to economic variables as in (10). The surplus may also react to inflation and other variables with parallel policy rules. I will shortly write down such rules in a more interesting model.

But this frictionless model is Fisherian: A rise in interest rates, with no change in surpluses, produces a rise in expected inflation one period later via \( i_t = E_t \pi_{t+1} \), and it produces no change in current inflation \( \Delta E_{t+1} \pi_{t+1} = 0 \). The frictionless new-Keynesian counterpart with \( \phi > 1 \) can produce a negative inflation response, by selecting instead an equilibrium with \( \Delta E_{t+1} \pi_{t+1} < 0 \). It implicitly assumes a coincident fiscal contraction, as (7) still holds, achieved by passive fiscal authorities. The fiscal theory of monetary policy can produce the same response, but would call it a coordinated fiscal and monetary shock.

### 2.3 Long-term debt

Now, add long-term debt with a geometric maturity structure, keeping for now a constant real interest rate and flexible prices. This model can generate lower inflation when the nominal interest rate rises, with no change in surpluses.

The model consists of the Fisher equation (6)

\[
i_t = E_t \pi_{t+1},
\]

and (5), which simplifies to

\[
\sum_{j=0}^{\infty} \omega^j \Delta E_{t+1} \pi_{t+1+j} = - \sum_{j=0}^{\infty} \Delta E_{t+1} s_{t+1+j}
\]

in place of (7). With long-term debt \( \omega > 0 \), shocks to the present value of surpluses in (12) now correspond to a change in the weighted average of current and expected future inflation.

When surpluses do not move, (12) introduces an important link between changes in expected inflation at different dates. Consider a persistent monetary policy shock – a persistent positive change in \( i_t = E_t \pi_{t+1} \), starting at \( i_1 = E_1 \pi_2 \), with no change in fiscal policy. From (12),

\[
\Delta E_1 \pi_1 = - \sum_{j=1}^{\infty} \omega^j \Delta E_1 \pi_{1+j}.
\]

If the terms \( \Delta E_1 \pi_{1+j} \) for \( j \geq 1 \) are, on average, positive, then \( \Delta E_1 \pi_1 < 0 \). In this way, with long-term debt, this positive persistent monetary policy shock induces a negative initial inflation response, which briefly overturns the otherwise Fisherian properties of this frictionless model.
This observation boils down a large effort in more complex models to produce a negative inflation response in a rational expectations model, without assuming a contemporary fiscal contraction (Sims (2011), Cochrane (2017b), Cochrane (2018)). The mechanism does not require sticky prices.

For example, suppose the central bank creates a monetary disturbance that follows an AR(1),

\[ i_t = \rho i_{t-1} + \varepsilon_t \]

with no change in surpluses. The inflation response to a unit shock \( \varepsilon_1 = 1 \) is higher expected inflation:

\[ \Delta E_1 \pi_{1+j} = \Delta E_1 i_j = \rho^{j-1}; \quad j = 1, 2, 3... \]

However, from (13), the impact effect of a higher interest rate is negative:

\[ \Delta E_1 \pi_1 = -\frac{\rho \omega}{1 - \rho \omega}. \]

Looking at (12), with short-term debt \( \omega = 0 \), a fiscal shock leads immediately to inflation at time 1. If the central bank chooses to follow that event with higher interest rates, the inflation will persist, but that is an independent choice and has no effect on immediate inflation \( \Delta E_1 \pi_1 \).

With long term debt \( \omega > 0 \), a fiscal shock may give rise to a persistent smaller rise in inflation, or even a rise only in future expected inflation with no current inflation at all. Contrary to the impression one gets with short-term debt models, then, fiscal theory does not just describe one-period price level shocks. The persistent inflation slowly devalues outstanding long-term debt. Since monetary policy controls expected inflation, monetary policy can now choose to smooth forward the fiscal shock, or not to do so.

As we shorten the time interval, and then move to continuous time without price level jumps, the effective \( \omega \) rises, and the instantaneous inflation term drops out altogether. All fiscal shocks now correspond to changes in expected future inflation, with monetary policy controlling the time path.

We get useful intuition by bringing bond returns back in to the analysis, expressing this model’s fiscal condition as the pair (3)-(4), which specialize to

\[ \Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} r_{t+1} = -\sum_{j=0}^{\infty} \Delta E_{t+1} s_{t+1+j} \]
\[
\Delta E_{t+1} r^n_{t+1} = - \sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \pi_{t+1+j}.
\]

(16)

With short-term debt \(\omega^j = 0\) there is no surprise bond return \(\Delta E_1 r^n_{t+1} = 0\). A price-level jump is the only way to devalue short-term debt. With long-term debt, a decline in bond prices and negative return \(r^n_{t+1}\) can reduce the value of long-term bonds, to bring the value of debt back in line with the present value of surpluses. And, with constant real returns, the decline in bond prices must come from expected future inflation, (16). Thus, a rise in expected future inflation can devalue long-term bonds, both when they are eventually redeemed with nominal dollars, and in their time-\(t+1\) market value. Expression (16) essentially marks the future inflation of (12) to market, to produce a present value identity (15) at time \(t + 1\).

Likewise, when the central bank raises interest rates and thereby expected future inflation, with short-term debt and no change in surpluses, this action has no effect on nominal bond prices, and therefore no effect on current inflation in (15). With long-term debt, when the central bank raises expected future inflation, that lowers long-term bond prices and the bond return \(r^n_{t+1}\). With no change on the right hand side of (15), immediate inflation must decline.

### 2.4 Fiscal policy

To understand the fiscal policy specification in a simple environment, let fiscal policy follow

\[
s_{t+1} = \alpha v^*_t + u^s_{t+1}
\]

(17)

\[
\eta v^*_t = v^*_t - s_{t+1}
\]

(18)

\[
\rho v^*_t = v^*_t - (E_t \pi_{t+1} - \pi_t) - s_{t+1}
\]

(19)

\[
u^s_{t+1} = \rho^s u^s_t + \epsilon^s_{t+1}
\]

(20)

Equation (19) is the debt evolution equation (1), with one period debt so \(i_t = r^n_{t+1}\), the Fisher equation \(i_t = E_t \pi_{t+1}\) and a constant interest rate.

The \(v^*\) term in the surplus policy rule (17) and (18) is a central innovation. This specification gives us a fiscal policy that is active, and rules out multiple equilibria, but nonetheless partially pays off debts accumulated from past deficits.

As one way to see how (18) works, compare it to (19). The surplus responds to a version of debt in which the ex-post real return is replaced by the expected real return, and the debt growth rate \(\rho^{-1}\) is increased to \(\eta^{-1}\). The first modification means that the surplus does not respond to unexpected and thus multiple equilibrium inflation. But \(v^*\) still accumulates past deficits, so surpluses do rise to pay off debts accumulated from past deficits. A fiscal policy rule that uses
v* rather than v mirrors different reactions of the interest rate target to equilibrium π* and off-equilibrium π − π* movements in (8).

The second modification, η > ρ, allows the government to partially repay debt and partially inflate. With η = ρ, all debts are paid off, ΔE_{t+1} \sum_{j=0}^{\infty} s_{t+1+j} = 0 in response to any fiscal shock, and there is no unexpected inflation in this model. With η > ρ, part of a surplus shock is repaid by future surpluses, and part by unexpected inflation. Intuitively, with η > ρ, we could write (35) as

\[ \rho + (\eta - \rho) v^*_t = v^*_t - s_{t+1}. \]

The variable v^*_t+1 is what the value of the debt would be if someone came along with an extra surplus (η − ρ)v^*_t+1. Surpluses react as if that gift had happened, i.e. they react less than fully to the actual deficit-induced rise in debt.

To see these points more explicitly, substitute (17) into (18), to write the v* process as

\[ v^*_t+1 = - \frac{1}{\eta + \alpha} \frac{1}{1 - \frac{1}{\eta + \alpha} L} u^*_t+1. \]  

Substituting (21) back into (17) we obtain the univariate moving average representation of the surplus process

\[ s_{t+1} = a(L)u^*_t+1 = \left( 1 - \alpha \frac{1}{1 - \frac{1}{\eta + \alpha} L} \right) u^*_t+1 = u^*_t+1 - \alpha \left( \frac{1}{\eta + \alpha} \right) u^*_t+1 - \alpha \left( \frac{1}{\eta + \alpha} \right)^2 u^*_t - \alpha \left( \frac{1}{\eta + \alpha} \right)^3 u^*_t-1... \]  

A deficit shock, a negative u^*_t, is followed by a string of small positive surpluses, which pay back some of the debt. We can also write (22) as an ARMA(1,1):

\[ a(L) = \frac{\eta}{\eta + \alpha} \left( 1 - \frac{1}{\frac{\eta}{\eta + \alpha} L} \right). \]

Adding the ARA(1) shock \( u^*_t+1 = (1 - \rho^s L)^{-1} \varepsilon^s_t+1 \) adds additional dynamics, producing an extended period of deficits followed by surpluses.

In (22) and (23), we see that we can also regard the pair (17) and (18) as a way to write compactly and intuitively a surplus process with a S-shaped moving average representation, in which a government running deficits promises future surpluses in order to borrow and not just devalue outstanding debts. In this interpretation, the variable v^*_t is just a latent state variable without further economic meaning. Surpluses do not “react to” v^*_t per se, they react to the past
deficits that \( v_t \) summarizes. We could just write the moving average (22) or (23) as the surplus process, with no \( v^* \). Introducing \( v^*_t \) writes an otherwise complex and potentially non-invertible MA process in a VAR(1) form, which is convenient for the usual VAR(1)-based matrix model solution technique.

The total response of the surplus \( \sum_{j=0}^{\infty} \rho^j \Delta E_{t} s_{1+j} \) to a unit transitory shock \( u^*_1 = 1, u^*_2 = u^*_3 = \ldots = 0 \) (i.e. with \( \rho^s = 0 \)) is given by \( a(\rho) = \sum_{j=0}^{\infty} \rho^j a_j \). (With persistent disturbances \( \rho^s > 0 \) the response is \( a(\rho)/(1 - \rho^s \rho) \).) From (22) or (23), we have

\[
a(\rho) = \frac{\eta - \rho}{\eta - \rho + \alpha}.\]

For \( \eta = \rho \), we have \( \alpha(\rho) = 0 \) all debts are repaid, and there are no inflationary surplus shocks, \( \Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j s_{t+1+j} = 0 \), in response to any (even persistent) shock \( u^*_t \).

Raising \( \eta \), and allowing \( \alpha(\rho) > 0 \) lets us model a surplus process in which the government promises to pay back only \( 1 - \alpha(\rho) \) part of the deficit, and let unexpected inflation soak up the rest \( \alpha(\rho) \) by devaluing outstanding bonds.

As \( \eta \to \infty \), \( v^*_t \to 0 \), and \( \alpha(\rho) \to 1 \). In this limit, we recover the usual sort of AR(1) surplus process, in which all of a surplus shock is financed by inflating away current debt.

To see the financing point, take the innovation of the debt accumulation equation (19),

\[
\rho \Delta E_{t+1} v_{t+1} = -\Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} s_{t+1}
\]

and consider the unexpected inflation identity (3) which simplifies in this case to

\[
\Delta E_{t+1} \pi_{t+1} = -\sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} s_{t+1+j} = a(\rho) u^*_t.
\]

When \( \eta = \rho \) and \( a(\rho) = 0 \), a negative surplus shock \( \Delta E_{t+1} s_{t+1} \) has no effect on inflation in (24), so it raises the value of debt \( v_{t+1} \). The government promises future surpluses to pay off additional debt. Therefore, it raises real resources by debt sales and the value of debt increases. When \( \eta \to \infty \) and \( a(\rho) = 1 \), then the transitory shock \( u^*_t \) leads only to unexpected inflation. The two terms on the right hand side of (24) cancel. The entire deficit is financed by devaluing outstanding bonds through unexpected inflation. The value of debt does not change at all, as there are no future surpluses.

In this schema \( a(\rho) \) between 0 and 1 offers us a model in which government meets a fiscal shock in part by borrowing, promising future real surpluses, and raising the value of debt, and in part by inflating away outstanding debt. This is surely the empirically relevant case. One
can see many avenues for generalization, in particular a time- or state-dependent split between inflation and future surpluses.

Next, examine the moving averages (21), (22). The condition for $v^*$ to converge, in (21), is

$$\frac{1}{\alpha + \eta} < 1. \quad (25)$$

Since $\eta > \rho$, the usual condition $\alpha > 1 - \rho$ is sufficient. Unstable $v^*$ is technically allowable, but it is cumbersome to allow just one explosive eigenvector in the solution method, so $\alpha > 1 - \rho$ is a convenient choice. Typically, the coefficient $1/(\eta + \alpha)$ is a number close to but below one, so the tail of moving average coefficients in (22) is very long.

The numerator coefficient of the surplus moving average polynomial (23) is $1/\eta$. Thus for values of $\eta$ equal to or just above $\rho$, the coefficient $1/\eta$ can equal or exceed one. In this case the moving average representations (22) and (23) are not invertible. The moving average representation is always non-invertible when the government pays all its debts, $a(\rho) = 0$.

There is nothing theoretically wrong with a non-invertible moving average forcing process. But a non-invertible representation means that we cannot measure the $s_t$ process from data on the history of $s_t$.

In sum them, viewed either in the form (22) or (23), this surplus process is of a type common, but all too frequently mistreated, in macroeconomics and finance, especially when we are interested in its long-run properties. In the form (22), we see a big movement in one direction, followed by a long string of offsetting movements in other direction. In (23) we see the classic ARMA(1,1) with nearly-canceling and near-unit coefficients. The difficulty of measuring the surplus response of this model is good news. It means, do not be too quickly discouraged by empirical work on surpluses that does not treat these subtleties with care.

We can understand specification (17) into (18) more deeply by comparing them with the corresponding standard specification of fiscal policy. Write instead of (17)-(20),

$$s_{t+1} = \gamma v_{t+1} + u_{t+1}^s \quad (26)$$

$$\rho v_{t+1} = v_t - \Delta E_{t+1} \Delta_{t+1} - s_{t+1} \quad (27)$$

$$u_{t+1}^s = \rho^s u_t^s + \varepsilon^s - t + 1. \quad (28)$$

Now surpluses respond to the value of the debt $v_t$, making no distinction between a rise in the value of debt coming from unexpected deflation and a rise coming from past deficits.

In this setup, any $\gamma > 1 - \rho$ (or, actually, $\gamma > 0$) results in a passive fiscal policy (Canzoneri,
Cumby, and Diba (2001)). To see this fact, substitute (26) into (27),

\[(\rho + \gamma) v_{t+1} = v_t - \Delta E_{t+1} \pi_{t+1} - u^s_{t+1}\]

and iterating forward

\[E_{t+1} v_{t+T} = \frac{1}{(\rho + \gamma)^T} (v_t - \Delta E_{t+1} \pi_{t+1}) - \sum_{j=0}^{T-1} \frac{1}{(\rho + \gamma)^{T-j}} E_{t+1} u^s_{t+1+j}.\]

The last term converges, since \(u^s_t\) is stationary. Thus if \(\gamma > 0\), the transversality condition \(\lim_{T \to \infty} \rho^T E_{t+1} v_{t+T}\) holds for any unexpected inflation \(\pi_{t+1} - E_t \pi_{t+1}\). We usually impose a slightly more stringent condition that debt itself is stationary, or “local” equilibria. Now if \(\gamma > 1 - \rho\), \(\lim_{T \to \infty} E_{t+1} v_{t+T} = 0\). Loosely, if you pay back any amount of outstanding debt, then debt grows at less than the interest rate. And if you pay the interest, plus any amount, then debts eventually vanish. So, even a small \(\gamma\) lets any value of unexpected inflation break out. It removes the extra forward-looking root from the government debt accumulation equation that we need for a determinate solution.

Great, so let’s set \(\gamma = 0\). But if we do that, and leave the shock as an AR(1), then the government never pays back any debts, including those accumulated from past deficits. All deficits are financed by unexpected inflation. And we obtain the strikingly counterfactual predictions that we should see inflation with current deficits, that current deficits do not result in larger values of the debt (Canzoneri, Cumby, and Diba (2001)), and that debt sales raise no revenue but just drive down bond prices along a unit-elastic demand curve.

To see the first prediction, just consider the unexpected inflation identity, with \(s_{t+1} = u^s_{t+1}\). We have

\[\Delta E_{t+1} \pi_{t+1} = - \sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} s_{t+1+j} = - \frac{1}{1 - \rho \rho^s} \varepsilon^s_t.\]

We can, and will, generalize the identity to include discount rate terms, but it’s unfortunate to start with so sharply counterfactual a prediction. To see the second prediction, consider (24),

\[\rho \Delta E_{t+1} v_{t+1} = -\Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} s_{t+1} = \frac{\rho \rho^s}{1 - \rho \rho^s} \varepsilon^s_t.\]

A higher surplus raises the value of debt. In reality, surpluses help to pay down debts. Well, if surpluses did indeed follow an AR(1), then higher surpluses today mean higher surpluses tomorrow.

Finally, if government does not promise higher surpluses when it sells debt to try to fund a
deficit, it raises no revenue from the debt sale. The debt sale is like a share split, which just drives down nominal bond prices. The fact that debt sales raise revenue proves that investors believe additional surpluses will follow today’s deficits. The point of this paper is that this observation does not instantly prove that fiscal policy is passive.

Thus, to date, using fiscal policy of the form (26)-(28), we are faced with an unpleasant choice: either abandon active fiscal policy, or sign on to a model that, at least in this simple specification, starts out with dramatically counterfactual predictions, and also lacks the commonsense mechanism that some deficits, sometimes, are paid for by borrowing real resources, and promising future surpluses to pay back the debt.

The answer, of course, is that surpluses don’t follow an AR(1). We can resolve the conundrum either directly, with $\gamma = 0$ and a $u^s$ process with an S-shaped moving average, or, as I did above, by thinking of the government as reacting to past deficits and real interest rates in $v^*$ but not to unexpected inflation as well, in $v$. In doing so either way we preserve active policy: (27) explodes forward at the rate $\rho^{-1}$, so only one value of unexpected inflation is an equilibrium.

The passive-fiscal model with $\eta = \rho$ in (26)-(27) also produces a non-invertible representation. One cannot measure the surplus process with an autoregression that excludes the value of debt. One can estimate the system (26)-(27), if one includes the value of debt $v_t$ in the VAR. Many VARs have been run incorrectly that exclude the value of debt, and one may be tempted to do so in a misguided attempt to test the present value identity.

To see this point, we can proceed as above, and write

$$s_{t+1} = \frac{\rho}{\rho + \gamma} \left( \frac{1}{1 - \frac{1}{\rho + \gamma} L} \right) u^s_{t+1} - \left( \frac{\gamma}{1 - \frac{1}{\rho + \gamma} L} \right) \epsilon^\pi_{t+1}.$$

With $\rho \leq 1$, $1/\rho \geq 1$ so the numerator of the first term is non-invertible. We can write the system (26)-(27) however as

$$s_{t+1} = \frac{\gamma}{\rho + \gamma} v_t - \frac{\gamma}{\rho + \gamma} \epsilon^\pi_{t+1} + \frac{\rho}{\rho + \gamma} u^s_{t+1}$$

$$\rho v_{t+1} = v_t - s_{t+1} - \epsilon^\pi_{t+1}.$$

We can estimate these equations by OLS.
3. A model with sticky prices and policy rules

Finally, I add sticky prices and policy rules. I use the standard new-Keynesian intertemporal substitution and Phillips curves,

\[ x_t = E_t x_{t+1} - \sigma (i_t - E_t \pi_{t+1}) \]  
\[ \pi_t = \beta E_t \pi_{t+1} + \kappa x_t. \]  

Eliminating the output gap \( x_t \) from (29)-(30), we have

\[ \beta E_t \pi_{t+2} - (1 + \beta - \sigma \kappa) E_t \pi_{t+1} + \pi_t = \sigma \kappa i_t. \]  

We can write this equation that expected inflation \( E_t \pi_{t+1} \) is a two-sided exponentially-weighted moving average of the interest rate \( i_t \),

\[ \pi_{t+1} = \frac{\sigma \kappa}{\lambda_1 - \lambda_2} \left[ i_t + \sum_{j=1}^{\infty} \lambda_1^{-j} i_{t-j} + \sum_{j=1}^{\infty} \lambda_2^{j} E_{t+1} i_{t+j} \right] + \sum_{j=0}^{\infty} \lambda_1^{-j} \Delta E_{t+1-j} \pi_{t+1-j} \]  

with weights given by the roots of the lag polynomial (31) (Cochrane (2018) p. 165), plus an exponentially decaying transient response to shocks. This formula naturally generalizes the Fisher equation (6) \( E_t \pi_{t+1} = i_t \) and hence \( \pi_{t+1} = i_t + \Delta E_{t+1} \pi_{t+1} \). Therefore, monetary policy can still determine expected inflation. It just takes a more complex interest rate path to give any particular expected inflation path.

The model consists of IS and Phillips equations (29), (30), fiscal and monetary policy rules, and the evolution of the value of government debt,

\[ i_t = \theta_\pi \pi_t + \theta_\pi x_t + u^i_t \]  
\[ s_t = \theta_\pi \pi_t + \theta_\pi x_t + \alpha v^s_t + u^s_t \]  
\[ \eta v^s_{t+1} = v^s_t + i_t - E_t \pi_{t+1} - s_{t+1} \]  
\[ \rho v_{t+1} = v_t + r^n_{t+1} - \pi_{t+1} - s_{t+1} \]  
\[ E_t r^n_{t+1} = i_t \]  
\[ r^n_{t+1} = \omega q_{t+1} - q_t \]  
\[ u^i_{t+1} = \rho^i u^i_t + \varepsilon^i_{t+1} \]  
\[ u^s_{t+1} = \rho^s u^s_t + \varepsilon^s_{t+1} \]
These are the same equations as the standard new-Keynesian model, with a set of implicit fiscal equations spelled out. I parameterize and solve the model with active fiscal rather than active monetary policy. The resulting model can stand as a benchmark fiscal theory of monetary policy, parallel to the standard three-equation new-Keynesian model.

Equation (33) is the monetary policy rule. Equation (34) is the fiscal policy rule. Both rules potentially react to inflation and output. I allow surpluses to respond to the output gap, as they do, with procyclical tax receipts and countercyclical stabilizers and stimuli. I allow surpluses to respond to inflation as well, which one can view either as a policy decision or imperfect indexation. As detailed in the last section, we either may regard $v^*$ as a way of implementing a policy that reacts to debts accrued from past deficits and real returns, but not to arbitrary unexpected inflation, or we can regard it as latent variable that allows us to write a S shaped moving average in the VAR(1) form that’s easy to solve by standard methods. Equation (35) specifies that $v^*$ and hence surpluses respond to increases in the value of debt that come from higher real interest rates, which with the expectations hypothesis in this model is the expected return on government bonds. Surpluses still do not respond to unexpected inflation. One can generalize (35) to include a specific value of unexpected inflation, if the government wishes to deliberately allow such inflation.

Equation (36) tracks the evolution of the real value of debt, from (1). Equation (37) is the bond pricing equation, using the expectations hypothesis that expected returns on bonds of all maturities are the same. Equation (38) relates the return on the government bond portfolio to its price $q_t$. Equations (39)-(40) allow persistence in monetary and fiscal policy disturbances.

The IS and Phillips curves (29)-(30) leave two undetermined expectational errors, needing two forward-looking roots to give a unique equilibrium. As usual, they have one forward and one backward-looking root, so we need one extra forward-looking root. In active-money new-Keynesian models $\theta_{\pi i} > 1$ (roughly speaking) generates the additional explosive root. I specify passive monetary policy with $\theta_{\pi i} < 1$. (With more complex specifications, one can create a passive-money model in which regressions of interest rates on inflation have a coefficient greater than one. As this model is not elaborated to be empirically realistic in its other equations, especially the IS and Phillips curves (29)-(30), I do not pursue that complication here.)

With short-term debt, we would have $r^n_{t+1} = i_t$, and the combination (34)-(36) would provide the extra explosive or unit root. Long-term debt adds another expectational error, (37), but one more unstable root in (38). Together (37)-(38) solve forward to

$$q_t = -E_t \sum_{j=1}^{\infty} \omega^{j-1} i_{t+j-1},$$
the expectations hypothesis that long-term bond prices reflect an average of future short-term interest rates.

The Appendix documents the algebra for solving the model in the standard way.

4. Responses

I present four sets of response functions. I start with responses to fiscal $u^s$ and monetary $u^i$ shocks with no policy responses $\theta = 0$. Then I add the policy responses. The first set of calculations is less realistic, but it helps to understand the model mechanisms – what responses are due to the economics of the model, rather than to endogenous policy reactions.

4.1 Fiscal shocks, no policy rules

Figure 1: Responses of the sticky-price long-term debt model, with no policy rules, to AR(1) surplus shock. Parameters are $\rho = 1$, $\theta = 0$, $\omega = 0.7$, $a(\rho) = 0.2$, $\sigma = 0.5$, $\kappa = 0.5$, $\rho^i = 0.9$, $\rho^s = 0.7$, $\alpha = 0.2$.

Figure 1 presents the response of this model to an AR(1) fiscal policy disturbance $u^s_t$, in the case of no policy rules $\theta = 0$. The “Fiscal, no $\theta$ rules” row of Table 1 presents the terms of the unexpected inflation decomposition (3), the bond return decomposition (4), and the expected inflation decomposition (5) for these responses. I pick the parameters to clearly illustrate mech-
\[
\Delta E_1 \pi_1 = - \Delta E_1 r_1^n = - \sum_{j=0}^{\infty} \Delta E_1 s_{1+j} + \sum_{j=1}^{\infty} \Delta E_1 \left( r_{1+j}^n - \pi_{1+j} \right)
\]

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<th>(-\Delta E_1 r_1^n)</th>
<th>(-\sum_{j=0}^{\infty} \Delta E_1 s_{1+j})</th>
<th>(+\sum_{j=1}^{\infty} \Delta E_1 \left( r_{1+j}^n - \pi_{1+j} \right))</th>
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Table 1: Terms of the inflation decomposition and bond-return decomposition for sticky-price model response functions.

Anisms, without attempting a serious match to data.

With neither monetary policy shock nor rule, the interest rate \(i\) and therefore long-term nominal bond return \(r^n\) do not move. Inflation rises and decays with an AR(1) pattern. This is already important news: Fiscal shocks result in drawn-out inflation, not a one-period price-level jump. The drawn-out inflation here is entirely the effect of sticky prices. It reflects the last term of (32), the exponentially decaying response to a shock in a sticky price model. As one reduces price stickiness, increasing \(\kappa\), the inflation response approaches the one-period price level jump of the flexible price model. This response is the same for any bond maturity \(\omega^j\). Since interest rates and bond returns do not move, maturity structure effects cannot make a difference.

Since inflation rises persistently, why don't nominal long-term bond prices and the return \(r^n\) fall? Since the interest rate does not move, the real rate falls exactly as inflation rises. Inflation and real rates naturally offset then, leaving no effect on nominal bond prices. The lower real discount rate (0.39) matching higher inflation (0.39) in the bond return decomposition shown in
the second panel of Table 1 shows this behavior.

Output rises mirroring the path of inflation, following the forward-looking Phillips curve that output is high when inflation is declining.

The surplus $s_t$ and the AR(1) surplus disturbance $u^s_t$ are not the same. The surplus initially declines, but deficits raise the $v^*$ latent variable, which accumulates past deficits. A long string of small positive surplus responses on the right side of the graph then partially repays the incurred debt.

This graph warns us of the empirical challenges ahead, and against many apparently easy rejections of fiscal theory. It would be hard to distinguish the surplus $s$ from the AR(1) disturbance $u^s$ in the data, as they differ only in the long run. The surplus seems to respond to the value of debt $v_t$ though it does not do so. Such an apparent response does not indicate passive fiscal policy.

The “Fiscal, no $\theta$ rules” row in the top panel of Table 1 presents the terms of the unexpected inflation decomposition (3) for these responses. What are the fiscal roots of this inflation? The 0.50% inflation shock corresponds to an even larger, 1.33% decline in the sum of future surpluses. However, since there is inflation but no change in nominal rates, real rates decline following the shock, which raises the value of debt and is therefore provides a 0.82% deflationary force. This endogenous decline in discount rate, brought on by sticky prices, buffers the effect of the surplus shock on inflation.

The 1% surplus shock leads to 1.33% decline in the sum of future surpluses, because surplus shocks are autocorrelated. If surpluses were an AR(1), then 1% shock $u^s$ would lead to a

$$\Delta E_1 \sum_{j=0}^{\infty} u^s_{1+j} = -\frac{1}{1 - \rho \rho^s} = -3.33\%$$

cumulative deficit. The fact that the the decline in surpluses is only 1.33% reflects the partial repayment promise, the response of surplus to $v^*_t$, and the parameter $a(\rho) = 0.2$, and the response of surpluses to output.

Summing it up in the bottom panel of Table 1, the fiscal shock produces 0.89% total inflation, spread through time. That inflation comes form a 1.33% cumulative fiscal shock (again, reflecting 3.33% shock, which sets off an offsetting partial repayment), offset by a 0.44% rise in discount rates.
Figure 2: Responses of the sticky-price long-term debt model, with no policy rules, to an AR(1) monetary policy shock. Parameters are $\theta = 0$, $\omega = 0.7$, $a(\rho) = 0.7$, $\sigma = 0.5$, $\kappa = 0.5$, $\rho^i = 0.9$, $\rho^s = 0.7$, $\alpha = 0.2$.

### 4.2 Monetary shocks, no policy rules

Figure 2 presents the response of variables in this model to an AR(1) monetary policy shock $u^i_t$, again with no policy rules to endogenous variables $\theta = 0$. With no policy rules, the nominal interest rate $i_t$ just follows the AR(1) shock process $u^i_t$.

Inflation $\pi$ declines, in contrast to the Fisherian model with one-period debt, and persistently, not just for one period, in contrast to the flexible-price long-term debt model of the last section. Sticky prices add persistence to the basic long-term debt mechanism outlined there.

Inflation does eventually rise to meet the higher nominal interest rate. This model remains Fisherian in the long run, a robust prediction of rational expectations models. Such models are stable going forward, so $i_t = E\pi$ is a stable steady state, and persistently higher interest rates eventually attract inflation.

Output also declines, again following the new-Keynesian Phillips curve in which output declines when inflation is rising.

Expected nominal returns $r^n_{j+1}$ follow the interest rate $i_j$, as this model uses the expectations hypothesis. That rise in expected returns and bond yields sends bond prices down, resulting in the sharply negative instantaneous bond return $r^b_t$. Subtracting inflation from these
nominal bond returns, the expected real interest rate rate and expected real bond return rises persistently.

Surpluses are not constant. Here, I define a monetary policy shock that holds constant the fiscal policy disturbance $u_t^s = 0$, but not surpluses $s_t$ themselves. Even though surpluses do not (yet) respond directly to inflation and output, surpluses respond to the increased value of the debt $v^*$ that results from higher real returns on government bonds. Here too, surpluses seem to respond to the value of the debt $v$, though in reality they are only responding to one component of that value $v^*$, which would lead one to incorrectly infer a passive fiscal policy. This is a first case of a warning which will echo several times – one must be careful to define just how one wants to orthogonalize monetary and fiscal policy in order to ask an interesting question.

The terms of the unexpected inflation decomposition (3) and of the bond return decomposition (4) in this response function are given by the “Monetary, no $\theta$ rules” rows of Table 1.

Start with the $\kappa = \infty$ flexible price (but still long-term debt) row. The negative nominal bond return $r_t^1 = -1.89\%$ is the same, as this depends only on the path of the nominal interest rate, which is the same. In this case, after one period of $-1.86\%$ deflation, there is a persistent inflation following the interest rate rise and $i_t = E_t \pi_{t+1}$. That persistent inflation is the entire cause of the lower long-term bond return, in the middle panel. And the lower bond price is the entire cause of the instantaneous deflation, in the upper panel, as neither surpluses nor discount rates change. This calculation reinforces the analysis of the flex-price long-term debt model in the last section. Higher nominal rates make long term bonds less valuable, but with an unchanged real value we must have a disinflation. As the bottom panel verifies the monetary policy shock only rearranges inflation. Total inflation is unchanged.

In the model with price stickiness, we see a different pattern. In the middle panel, the $-1.89\%$ bond return is the same, but it comes almost entirely from a rise in real rates, not from a rise in inflation. Price stickiness, and the resulting long period of disinflation with higher nominal rates, means the same nominal rate rise is primarily a real rate rise. In the bottom panel, there has been an overall disinflation (-1.08) fueled by a large rise in surpluses offset by a medium rise in discount rate.

In the top panel, that bond return has the same 1.89% deflationary effect. But now surpluses rise, by a large cumulative 2.63%, a large inflationary effect. Again, the surplus rise is an indirect result of higher real interest rates and the surplus rule that agrees to pay off rises in debt that come from real rates. But the same rise in real interest rates raises the discount rate for surpluses by a cumulative 3.28%, an inflationary effect. Overall, the discount rate wins, so disinflation $\Delta E_1 \pi_1$ is $-1.24\%$, not $-1.89\%$. 
In sum, the real interest rate effects of monetary policy with sticky prices deeply mediate the effects of monetary policy on inflation. Higher real interest rates discount surpluses at a higher rate, an inflationary force. Higher real interest rates lead here to higher surpluses, a deflationary force. Even in this simple example accounting for inflation involves multiple, and often countervailing fiscal forces.

4.3 Response to a fiscal shock with policy rules

Figure 3: Response to a fiscal policy shock in the sticky-price long-term debt model, with endogenous policy rules. Parameters add $\theta_{ix} = 1$, $\theta_{i\pi} = 0.5$, $\theta_{s\pi} = 0.5$, $\theta_{s\pi} = 0$.

Figure 3 presents the response to a fiscal shock, holding constant the monetary policy disturbance $u^t_i$ but now allowing both fiscal and monetary policy rules.

The instantaneous inflation is about half its previous value, but inflation is much more persistent. Endogenous policy responses smooth the inflationary effects of a shock, both directly and by smoothing inflation forward.

Monetary policy reacts to higher inflation and output by raising the nominal interest rate, which was constant before. Greater output gives also causes larger fiscal surpluses through the fiscal policy rule. Higher nominal interest rates also occasions a fall in bond prices, which soaks up some of the fiscal shock.

Table 1 quantifies these mechanisms. In the second row, instantaneous inflation is half its
previous value (0.23). The larger (less negative) surpluses contribute to lower inflation (−0.72), but the change in discount rates offsets that change pretty much entirely. The big difference is that of the $0.72 + 0.14 = 0.86\%$ fiscal inflationary force, $0.34\%$ is soaked up by a decline in nominal bond prices. In the lower panel, that negative return (−0.34) comes now entirely from future inflation. So we have even in this more complex model the simple picture alluded to above, that monetary policy smooths an inflationary fiscal shock forward, producing a protracted inflation that slowly devalues long-term bonds rather than a sharp price level jump.

This example produces the drawn out inflation of the 1970s, arguably the result of fiscal shocks, but not the lower output characteristic of stagflation. That failure is likely rooted in the nature of this Phillips curve.

4.4 Response to a monetary shock with policy rules

![Figure 4: Response to a monetary policy shock in the sticky-price long-term debt model, with endogenous policy rules. Parameters add $\theta_{ix} = 1$, $\theta_{i\pi} = 0.5$, $\theta_{sx} = 0.5$, $\theta_{s\pi} = 0$.](image)

Figures 3 and 4 plot responses to the monetary policy shock, but now adding policy rules,

\begin{align}
    i_t &= 0.5 \pi_t + 1.0 x_t + u_t^i \\
    s_t &= 0 \pi_t + 0.5 x_t + 0.2 v_t^s + u_t^s
\end{align}
Again, I choose the coefficients to make the graphs clear and to illustrate mechanisms, not in any attempt to match data or estimates. Table 1 includes the inflation and bond return decompositions, in the rows marked “yes θ rules.”

In Figure 4 we see that the policy rule pushes the interest rate $i$ below its disturbance $u^i$. Both inflation $\theta_{i\pi}$ and output $\theta_{ix}$ responses contribute to this lower interest rate. I held down the coefficient $\theta_{i\pi} = 0.5$, rather than a more traditional larger value, to keep the interest rate response from being negative, the opposite of the shock. Interest rates that go in the opposite direction from monetary policy shocks are a common feature in new–Keynesian models of this sort. (Cochrane (2018) p. 175 shows some examples.) But they are confusing, and my point here is to illustrate mechanisms. The interest rate then rises gradually, along with inflation, before settling down long past the right end shown in the figure. Long-term bonds again suffer a negative return on impact, as the interest rate does rise uniformly, but less than before.

The effect of monetary policy is also modified significantly by the fiscal policy rule, though there is no fiscal policy shock. Relative to Figure 2, the output decline following the monetary policy shock now leads to a large deficit. The surplus again rises eventually to partially, but not totally, pay off that debt.

The “Monetary, yes θ rules” rows of Table 1 again quantify these offsetting effects on inflation. Here, the overall surplus response is still positive though half of its value without endogenous monetary and fiscal policy responses. The smoother interest rate path, lower than its disturbance $u^s$, implies a lower bond return shock $\Delta E_1 r^m_1$, and the discount rate effect is also less than half its previous value. The negative bond return, coming from the largely positive response in nominal interest rates, now results from a combination of inflation and real interest rate effects. Inflation rises more quickly, so the weighted sum of future inflation is larger.

Just how one defines and orthogonalizes monetary and fiscal policy is a crucial but subtle matter. Here I define a monetary policy shock that does not affect the fiscal shock $u^s_t$. But monetary policy nonetheless has fiscal consequences: The fiscal rule responds to output, (potentially) to inflation, and to real-interest-rate-induced rises in the value of debt. This is not passive fiscal policy in the traditional definition, since it does not respond to multiple-equilibrium unexpected-inflation induced variation in the value of the debt. But it is a likely fiscal response to a monetary policy shock.

Should an analysis of the effects of monetary policy include these systematic fiscal policy responses? They are every bit as much human decisions as the fiscal policy shocks $u^s_t$. But they are predictable endogenous responses just like the private-sector decisions that are the whole point of an economic model.
I think yes, and that the Federal Reserve might like predictable endogenous responses of fiscal events, via output and inflation responses of the tax code and automatic stabilizers, to be part of a model’s assessment of the consequences of monetary policy. They may even want predictable actions of fiscal authorities, as in stimulus programs. But the issue is really just what do we – and the Federal Reserve – find an interesting question. The main point in this paper is that one can include such endogenous reactions or policy rules. Or not.

Since fiscal shocks often have a more exogenous air about them, including the likely endogenous monetary policy response makes even more sense. Adding a likely Fed shock – a deliberate $u_{t+1}$ taken in response to the event causing a fiscal policy shock – might be sensible too. Again, we just need to be careful about what questions are interesting, and the point is that one can ask such questions.

4.5 The way forward

This model is still simple and unrealistic. I advance it to show what can be done.

One hungers, of course, for a model that one can bring to data, estimate parameters, and formally match impulse-responses to structural and policy shocks.

While everyone knows these IS and Phillips curves are wrong, a standard more successful alternative has not emerged. It is likely that the form of these curves that fits best will be different under a fiscal equilibrium than it is in a standard new-Keynesian model. One wishes, in the end, something like a Smets and Wouters (2007), or Christiano, Eichenbaum, and Evans (2005), adapted to fiscal theory as I adapted the textbook new-Keynesian model above. Eventually one wants a more ambitious model incorporating habits or other dynamic preferences and investment adjustment costs in order to produce hump-shaped response functions, heterogeneity, variation in risk premia, labor market and investment frictions, the latest in financial frictions, zero bounds, and so forth. A major point of this paper is that one can construct such models, and quite easily from a technical standpoint. But finding the right model is not so easy, as that specification search has not been so easy for standard new-Keynesian models.

My monetary policy rule is simplistic, needing at least lags and a zero bound, plus matching policy rule regressions in data. Estimating the fiscal policy rule is a challenge of similar order, not yet started, and made even more challenging by the fact that any sensible rule, such as this one, has subtle but crucial long-run responses, or a latent state variable.

Like the rest of the model, this surplus process can and should be generalized towards realism in many ways. In particular, the $v^*$ process can respond to one particular value of unexpected inflation, rather than the strict zero-inflation target here. News about future surpluses
and historical episodes are likely not well modeled by AR(1) shocks to the disturbance $u_t^s$. The choice to finance deficits by inflating existing debt vs. borrow against future surpluses is likely to change over time and in response to state variables as well. One may wish to add the option for explicit default and the choice of default vs. inflation. And identifying and measuring the fiscal policy rule from equilibrium data may prove just as challenging as the monetary policy rule. But each of these steps is also an obvious, but unexplored opportunity.
References


Online Appendix to “A Fiscal Theory of Monetary Policy with Partially Repaid Long-Term Debt”

This Appendix sets out the algebra to solve the model (29)-(40). I express the model in the form

\[ Ay_{t+1} = By_t + C\varepsilon_{t+1} + D\delta_{t+1} \]  

(43)

where \( y \) is a vector of variables, \( \varepsilon \) are the structural shocks, and \( \delta \) are expectational errors in the equations that only tie down expectations. We eigenvalue decompose the transition matrix \( A^{-1}B \), we solve unstable roots forward and stable roots backward to determine the expectational errors \( \delta \) as a function of the structural shocks \( \varepsilon \). Then, we can compute the impulse-response function. I use notation \( \gamma \equiv 1 - a(\rho) \) and \( \rho = 1 \). Then we have \( \eta = 1 + (1 - \gamma)/\gamma\alpha \).

First, eliminate redundant variables to write (29)-(40) as

\[
\begin{align*}
    x_{t+1} + \sigma\pi_{t+1} = x_t + \sigma \left( \theta_{ix}\pi_t + \theta_{ix}x_t + u_t^i \right) + \delta_{t+1}^x + \sigma\delta_{t+1}^\pi \\
    \pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}\varepsilon_t + \delta_{t+1}^\pi \\
    v_{t+1}^* + (\theta_{sx}\pi_{t+1} + \theta_{sx}x_{t+1} + \alpha v_{t+1}^* + u_{t+1}^s) = \left( \theta_{ix}\pi_t + \theta_{ix}x_t + u_t^i \right) - \left( \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}\varepsilon_t \right) + v_t^* \\
    v_{t+1} - (\omega q_{t+1} - q_t) + \pi_{t+1} + \left( \theta_{sx}\pi_{t+1} + \theta_{sx}x_{t+1} + \alpha v_{t+1}^* + u_{t+1}^s \right) = v_t \\
    \omega q_{t+1} = \left( \theta_{ix}\pi_t + \theta_{ix}x_t + u_t^i \right) + q_t + \omega\delta_{t+1}^q, \\
    or \\
    x_{t+1} + \sigma\pi_{t+1} = (1 + \sigma\theta_{ix}) x_t + \sigma\theta_{ix}\pi_t + \sigma u_t^i + \delta_{t+1}^x + \sigma\delta_{t+1}^\pi \\
    \pi_{t+1} = -\frac{\kappa}{\beta} x_t + \frac{1}{\beta}\pi_t + \delta_{t+1}^\pi \\
    \theta_{sx}x_{t+1} + \theta_{sx}\pi_{t+1} + \left( 1 + \frac{\alpha}{\gamma} \right) v_{t+1}^* + u_{t+1}^s = \left( \theta_{ix} + \frac{\kappa}{\beta} \right) x_t + \left( \theta_{ix} - \frac{1}{\beta} \right) \pi_t + v_t^* + u_t^i \\
    \theta_{sx}x_{t+1} + (1 + \theta_{sx})\pi_{t+1} + \alpha v_{t+1}^* + v_{t+1} - \omega q_{t+1} + u_{t+1}^s = v_t - q_t \\
    \omega q_{t+1} = \theta_{ix}x_t + \theta_{ix}\pi_t + q_t + u_t^i + \omega\delta_{t+1}^q.
\end{align*}
\]


In matrix notation (43),

\[
\begin{bmatrix}
1 & \sigma & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\theta_{xx} & \theta_{x\pi} & 1 + \alpha/\gamma & 0 & 0 & 1 \\
\theta_{xx} & 1 + \theta_{x\pi} & \alpha & 1 - \omega & 0 & 1 \\
0 & 0 & 0 & 0 & \omega & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_{t+1} \\
\pi_{t+1} \\
v_{t+1}^* \\
v_{t+1} \\
q_{t+1} \\
q_{t+1}^i \\
u_{t+1}^i \\
u_{t+1}^s \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & \sigma & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \theta_{xx} & \theta_{x\pi} & 1 + \alpha/\gamma & 0 \\
0 & 0 & \theta_{xx} & 1 + \theta_{x\pi} & \alpha & 1 - \omega \\
0 & 0 & 0 & 0 & \omega & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_t \\
\pi_t \\
v_t^* \\
v_t \\
q_t \\
u_t^i \\
u_t^s \\
\end{bmatrix}
+ 
\begin{bmatrix}
1 & \sigma & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho^i & 0 \\
0 & 0 & 0 & 0 & 0 & \rho^s \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{t+1}^i \\
\varepsilon_{t+1}^s \\
\delta_{t+1}^x \\
\delta_{t+1}^\pi \\
\delta_{t+1}^q \\
\end{bmatrix}
+ 
\begin{bmatrix}
\varepsilon_{t+1}^i \\
\varepsilon_{t+1}^s \\
\delta_{t+1}^x \\
\delta_{t+1}^\pi \\
\delta_{t+1}^q \\
\end{bmatrix}
+ 
\begin{bmatrix}
\delta_{t+1}^x \\
\delta_{t+1}^\pi \\
\delta_{t+1}^q \\
\end{bmatrix}
\end{array}
\]

Now, we solve the model as

\[
Ay_{t+1} = By_t + C\varepsilon_{t+1} + D\delta_{t+1}
\]

\[
y_{t+1} = A^{-1}By_t + A^{-1}C\varepsilon_{t+1} + A^{-1}D\delta_{t+1}
\]

\[
y_{t+1} = QAQ^{-1}y_t + Q^{-1}A^{-1}C\varepsilon_{t+1} + Q^{-1}A^{-1}D\delta_{t+1}
\]

\[
Q^{-1}y_{t+1} = \Lambda Q^{-1}y_t + Q^{-1}A^{-1}C\varepsilon_{t+1} + Q^{-1}A^{-1}D\delta_{t+1}
\]

\[
z_{t+1} = \Lambda z_t + Q^{-1}A^{-1}C\varepsilon_{t+1} + Q^{-1}A^{-1}D\delta_{t+1}
\]

Let \(G_f\) select rows with eigenvalues greater than one, and \(G_b\) select rows with eigenvalues less than one. For example, if the first and third eigenvalues are greater than or equal to one,

\[
G_f = \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \end{bmatrix}.
\]
Then, the \( z \) corresponding to eigenvalues greater than one must be zero so

\[
0 = G_f Q^{-1} A^{-1} C \epsilon_{t+1} + G_f Q^{-1} A^{-1} D \delta_{t+1}
\]

\[
\delta_{t+1} = - \left( G_f Q^{-1} A^{-1} D \right)^{-1} G_f Q^{-1} A^{-1} C \epsilon_{t+1}
\]

For this to work there must be as many rows of \( G_f \) as columns of \( \delta \), i.e. as many eigenvalues greater or equal to one as there are expectational errors. Substituting, we have the evolution of the transformed \( z \) variables, i.e. the impulse response function,

\[
z_{t+1} = \Lambda z_t + Q^{-1} A^{-1} \left[ I - D \left( G_f Q^{-1} A^{-1} D \right)^{-1} G_f Q^{-1} A^{-1} \right] C \epsilon_{t+1},
\]

and then the original variables from

\[
y_t = Q z_t.
\]

I include two small refinements. First, for computation it is better to force the elements of \( z_t \) that should be zero to be exactly zero. Machine zeros \((1e-14)\) multiplied by explosive eigenvalues eventually explode. Thus, I find the non-zero \( z \) only by simulating forward the nonzero elements of \( z \),

\[
G_b z_{t+1} = G_b \Lambda z_t + G_b Q^{-1} A^{-1} \left[ C - D \left( G_f Q^{-1} A^{-1} D \right)^{-1} G_f Q^{-1} A^{-1} C \right] \epsilon_{t+1}.
\]

Second, the consumer’s transversality condition tells us that debt \( v_t \) cannot explode. There is no reason to impose that the latent state variable \( v_t^* \) cannot explode or have a unit root. In solving the model for some parameter values it is important not to unwittingly impose that condition. The most obvious example occurs for passive fiscal policy, if \( s_t = \ldots + \alpha v_t + \ldots \), not \( s_t = \ldots + \alpha v_t^* + \ldots \). Then \( v_t^* + s_{t+1} + \ldots = v_t^* \) has a unit root (or explosive in the usual model with discounting), but the quantity \( v_t^* \) enters nowhere else in the model. We seem to get determinacy by adding a useless unit root variable.

Rather than \( \lim_{T \to \infty} E_{t+1} y_{t+T} = 0 \), we need to impose

\[
\lim_{T \to \infty} RE_{t+1} y_{t+T} = 0
\]
where $R$ is of the form

$$
R = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \ddots
\end{bmatrix},
$$

i.e. omitting the row of $y_t$ corresponding to $v_t^*$ (or any other variable that can explode). Then, rather than simply setting to zero the $z$ corresponding to unit and greater eigenvalues, we need to set only

$$
\lim_{T \to \infty} RQE_{t+1} z_{t+T} = \lim_{T \to \infty} RQ \Lambda^T z_{t+1} = 0.
$$

Denote by $\lambda_{<1}$ the eigenvalues less than one and $\lambda_{>1}$ the eigenvalues greater than one, and similarly for the corresponding $z$. We want, for example,

$$
\lim_{T \to \infty} Q^T \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_{<1} & 0 & 0 & 0 & 0 \\
0 & \lambda_{<1} & 0 & 0 & 0 \\
0 & 0 & \lambda_{>1} & 0 & 0 \\
0 & 0 & 0 & \lambda_{>1} & 0 \\
0 & 0 & 0 & 0 & \lambda_{>1}
\end{bmatrix}^T
\begin{bmatrix}
z_{<1} \\
z_{<1} \\
z_{>1} \\
z_{>1} \\
z_{>1}
\end{bmatrix} = 0.
$$

(The actual system is larger.)

Let $G_f^*$ denote a matrix with ones in the place of eigenvalues greater or equal to one and zeros elsewhere, for example,

$$
G_f^* = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

This is the matrix $G_f$ above with zero rows added back. A simple test whether this problem is occurring is whether the rank of $RQ G_f^*$ is the same as the rank of $Q G_f^*$, i.e. of $G_f^*$ itself since $Q$ is full rank. If that test succeeds, then we are not using the false condition that $v^*$ may not explode to set a linear combination of the $z$ to zero.
If that test fails, then in place of setting $G^*_f z_{t+1} = 0$, we set $RQG^*_f z_{t+1} = 0$, i.e.
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
z_{<1} \\
z_{<1} \\
z_{>1} \\
z_{>1} \\
z_{>1} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}.
\]
Express the matrix on the right hand side in row-echelon form, delete the rows with zeros, and proceed as before.