Research Article

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Optimal portfolio choice and stochastic volatility

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In this paper we examine the effect of stochastic volatility on optimal portfolio choice in both partial and general equilibrium settings. In a partial equilibrium setting we derive an analog of the classic Samuelson–Merton optimal portfolio result and define volatility-adjusted risk aversion as the effective risk aversion of an individual investing in an asset with stochastic volatility. We extend prior research which shows that effective risk aversion is greater with stochastic volatility than without for investors without wealth effects by providing further comparative static results on changes in effective risk aversion due to changes in the distribution of volatility. We demonstrate that effective risk aversion is increasing in the constant absolute risk aversion and the variance of the volatility distribution for investors without wealth effects. We further show that for these investors a first-order stochastic dominant shift in the volatility distribution does not necessarily increase effective risk aversion, whereas a second-order stochastic dominant shift in the volatility does increase effective risk aversion. Finally, we examine the effect of stochastic volatility on equilibrium asset prices. We derive an explicit capital asset pricing relationship that illustrates how stochastic volatility alters equilibrium asset prices in a setting with multiple risky assets, where returns have a market factor and asset-specific random components and multiple investor types. **Copyright © 2011 John Wiley & Sons, Ltd.**

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1. Introduction

This paper examines the effect of stochastic volatility on portfolio choice and equilibrium asset prices. Asset pricing models, both theoretical and empirical, routinely include stochastic volatility. However, it is difficult to evaluate what different volatility assumptions imply about economic behavior. In this paper we derive a stochastic volatility analog of the classic Samuelson [1]–Merton [2] partial equilibrium optimal portfolio result, with an explicit expression for the effective risk aversion under stochastic volatility. We refer to this measure as the volatility-adjusted risk aversion. In this partial equilibrium setting we provide comparative statics on how volatility-adjusted risk aversion changes with the distribution of the unknown volatility. We also examine the effect of stochastic volatility in a general equilibrium setting. We derive a stochastic volatility analog of the capital asset pricing model (CAPM) and illustrate how the distribution of volatility affects equilibrium asset prices through its effect on volatility-adjusted risk aversion in a setting with multiple risky assets and investor types.

To perform our analysis we derive an extension of Stein's lemma [3] that allows us to separate utility effects from random variable covariance effects. Rubinstein [4] and Huang and Litzenberger [5] illustrate the use of Stein's lemma for portfolio choice problems and Constantinides [6] provides a further discussion. The stochastic volatility version of Stein's lemma differs from the standard result by adding a proportionality factor based on a size-biased change of measure for the unobserved stochastic volatility. This enables us to explicitly define volatility-adjusted risk aversion which we call VARA. It differs from the standard risk aversion term by requiring an expectation taken over this size-biased measure. Using our extension of Stein's lemma we are able to show that the optimal risky asset allocation is separable as the expected excess return scaled by the return variance and VARA, analogous to the standard problem analyzed by Samuelson and Merton.

Our research contributes to the literature on optimal portfolio allocation by examining precisely how changes in stochastic volatility affect portfolio choice. Our work is most closely related to Coles *et al.* [7], who compared optimal portfolio choices and equilibrium payoffs in the presence of stochastic volatility to those without. They establish that individuals with exponential utility are effectively more risk averse and asset prices are lower in the presence of stochastic volatility. Our

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approach, with an explicit expression for VARA, allows for further comparative statics on how changes in the distribution of the unknown variance change the effective risk aversion under stochastic volatility. In particular, we show that a second-order stochastic dominating shift in the volatility distribution increases effective risk aversion, while this is not guaranteed for a first-order stochastic dominating shift in the volatility distribution. Our approach differs from recent work analyzing portfolio choice in continuous time, such as Liu [8], Chacko and Viceira [9], and Xia [10]. Our single period setting allows us to focus on the effects of stochastic volatility on effective risk aversion whereas in a continuous time setting the level of current volatility and the uncertainty of future volatility are the primary determinants of portfolio choice[‡] .

Our results also contribute to research that relates observations about asset markets to individual investor's behavior. One line of this research investigates the effect of asset return predictability on optimal portfolio allocation (see, for example, [11–14]). A related literature examines other empirical regularities, such as the equity premium puzzle, and infers the individual characteristics that would be consistent with these observations. Our results allow research in these areas to derive an analytic solution for the optimal portfolio allocation decision with parameter uncertainty in the covariance matrix with a separable, well-defined volatility-adjusted risk aversion parameter. This allows researchers to assess the economic behavior underlying different models of stochastic volatility and to investigate the sensitivity of the results to the specification of the volatility distribution.

The paper proceeds as follows. Section 2 introduces stochastic volatility and our measure of volatility-adjusted risk aversion. We then examine the effect of stochastic volatility in three different settings. First, we consider a partial equilibrium setting where a marginal investor takes the stock price as given. We derive an extension of Stein's lemma to stochastic volatility distributions and use it to find a general analog to the classic Samuelson–Merton optimal portfolio result. Second, in Section 3, we consider an investor who has constant absolute risk aversion, thus without complicating wealth effects. In this case we derive comparative statics for the effect of changes in the distribution of stochastic volatility on effective risk aversion of a representative investor. Finally, in Section 4, we endogenously derive market-clearing equilibrium stock prices when firms' future cash flows are affected by a single factor that exhibits stochastic volatility. We verify that the resulting equilibrium stock returns adhere to CAPM even when there is stochastic volatility. Section 5 provides a summary and discussion. All proofs are in the appendix.

2. Asset allocation and stochastic volatility

In this section we derive a closed-form expression for the optimal asset allocation under stochastic volatility. To do so, we present our version of Stein's lemma that applies to random variables with stochastic volatility. We begin, however, with the standard problem without stochastic volatility to introduce notation and provide a reference point.

In the individual's portfolio problem, an agent has initial wealth that can be invested in a risk-free bond or in a risky stock. The bond yields a known gross return of R_f at the end of the period. The risky stock has a stochastic gross return of R. We normalize the initial wealth to unity and denote the fraction of initial wealth invested in the risky stock as ω. Then final wealth is $W=(1-\omega)R_f+\omega R$. The agent's problem is to choose ω to maximize the expected utility of final wealth, or $\max_\omega E[U(W)]$. If U is twice differentiable, increasing and strictly concave in ω, the optimal allocation is characterized by the first-order condition:

$$E[U'(W)(R-R_f)]=0.$$

Applying the definition of covariance yields

$$\mathrm{Cov}[U'(W), R-R_f]+E[U'(W)]E[R-R_f]=0.$$

When the random variable R is normally distributed, Stein's lemma can be applied to derive a more informative expression. Stein's lemma equates the covariance of a function of normal random variables to the underlying covariance times a proportionality constant. More precisely, let X denote a normal random variable, $X\sim N(\mu,\sigma^2)$ with mean μ and variance σ^2 and let $g(X)$ be the differentiable function of X such that $E[|g'(X)|]<\infty$. Then $\mathrm{Cov}[g(X), X]=E[g'(X)]\sigma^2$. In the bivariate case for normal random variables (X, Y) Stein's lemma becomes $\mathrm{Cov}[g(X), Y]=E[g'(X)]\mathrm{Cov}[X, Y]$, see Stein [3].

[‡] *In continuous time the instantaneous volatility is known with certainty and only future volatility is stochastic. Thus the portfolio allocation problem in continuous time is driven primarily by the level of current volatility and the uncertainty of future volatility. In that our discrete time, single period approach abstracts from inter-temporal substitution and hedging demands, our setting is more appropriate to situations where the investment horizon is long relative to the change in volatility.*

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Applying this identity to the first-order condition yields

$$\omega E[U''(W)]\text{Var}[R]+E[U'(W)](E[R]-R_f)=0.$$

Hence, the optimal allocation $\omega^\star$ is

$$\omega^\star=\frac{1}{\gamma}\left(\frac{E[R]-R_f}{\text{Var}[R]}\right), \tag{1}$$

where γ is the agent's global absolute risk aversion: $\gamma=-E[U''(W)]/E[U'(W)]$. This is the well-known Samuelson [1] and Merton [2] result that the optimal portfolio weight is multiplicatively separable in risk aversion and the market price of risk.

2.1. *Stein's lemma for stochastic volatility*

We extend this result to the case where the distribution of the risky asset return has stochastic volatility. To do so we now develop an extension of Stein's lemma that applies to random variables with stochastic volatility. In general, a random variable X whose volatility is drawn from a probability density function $p(V)$ is said to exhibit stochastic volatility, V, if we can write $X|V\sim N(\mu,\sigma^2 V)$ where $V\sim p(V)$, $V\geqslant 0$, and $\sigma^2>0$[§] .

The distribution of outcomes X is

$$p(X)=\int_0^\infty p(X|V)p(V)\,\mathrm{d}V.$$

With stochastic volatility, the marginal distribution of X has heavy tails[¶] . Stein's lemma equates the covariance of a function of normal random variables to the underlying covariance times a proportionality factor. The critical difference between the Stein's lemma and our derivation is that the proportionality factor undergoes a change of measure that we denote by $q(V)$. This density comes from size-biasing the density of the volatility and is defined as

$$q(V)=\frac{V p(V)}{E[V]}, \tag{2}$$

where we assume that $0<E[V]<\infty$. We define $q(X)$ to be the induced marginal distribution of X given by this measure $q(V)$ on the volatility, namely

$$q(X)=\int_0^\infty p(X|V)q(V)\,\mathrm{d}V. \tag{3}$$

In general, size-biasing in this way causes the density $q(X)$ to have heavier tails than the original density $p(X)$.

Theorem 1 (*Stein's lemma for stochastic volatility*)
Let X be a random variable with a stochastic volatility so that $X|V$ is distributed $N(\mu,\sigma^2 V)$ and V has density $p(V)$ that is non-negative only for $V\geqslant 0$. Suppose that $0<E[V]<\infty$. Then, we have

$$\text{Cov}[g(X),X]=E^Q[g^{'}(X)]\text{Var}[X].$$

Further, if $(X,Y|V)$ are bivariate Normal random variables then

$$\text{Cov}[g(X),Y]=E^Q[g^{'}(X)]\text{Cov}[X,Y],$$

where E^Q is the expectation taken under the measure induced by size-biasing $q(V)=Vp(V)/E[V]$.

Proof
See Appendix A.

[§] *There is a slight abuse of notation as we use V to denote a random variable and $p(V)$ its density.*

[¶] *The class of stochastic volatility distributions has a number of applications in financial economics, mainly because it allows for heavy tails. The family of distributions includes the well-used t_ν-distribution, Exponential power [15], Stable [16], elliptical distributions [17], variance gamma distributions [18], and the Double Exponential and logistic distributions [19, 20].*

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2.2. Asset allocation under stochastic volatility

We can now derive the optimal portfolio choice when gross returns, R, exhibit stochastic volatility. Thus $R|V$ is distributed normally and $V \sim p(V)$, $V \geqslant 0$. Applying our theorem to the portfolio choice problem with stochastic volatility yields the identity $\mathrm{Cov}[U'(W), R-R_f] = E^Q[U''(W)]\mathrm{Var}[R]$. This can be substituted into the first-order condition, and proceeding as before we find that the optimal share of wealth allocated to the risky asset is

$$\omega^{\star}_{SV} = \frac{-E[U'(W)]}{E^Q[U''(W)]}\left(\frac{E[R]-R_f}{\mathrm{Var}[R]}\right). \tag{4}$$

We define the volatility-adjusted risk aversion, Γ, as

$$\Gamma = \frac{-E^Q[U''(W)]}{E[U'(W)]}, \tag{5}$$

where E^Q is the expectation taken under the measure induced by size-biasing the volatility distribution $q(V) = Vp(V)/E[V]$. Using our definition of VARA, we can simplify our expression for the optimal portfolio weight as

$$\omega^{\star}_{SV} = \frac{1}{\Gamma}\left(\frac{E[R]-R_f}{\mathrm{Var}[R]}\right). \tag{6}$$

By applying our extension of Stein's lemma we find that the optimal portfolio weight with stochastic volatility is again separable into risk aversion and market price of risk. We define Γ as the volatility-adjusted risk aversion, VARA, that characterizes the investor's risk preferences under stochastic volatility.

3. Absolute volatility-adjusted risk aversion

We now examine the effects of stochastic volatility on VARA in a partial equilibrium model where we assume that the share invested in the risky asset is exogenously given. To isolate risk aversion effects we consider an investor with exponential utility of final wealth, $U(W) = -\mathrm{e}^{-\gamma W}$, where $\gamma > 0$. In the absence of stochastic volatility the investor has constant absolute risk aversion, namely $-U''(W)/U'(W) = \gamma$. However, when we account for stochastic volatility, investors act as if their risk aversion is given by the VARA measure. To be clear about this particular setting, we refer to VARA in this special case as absolute VARA. Our first set of results compare effective risk aversion with and without stochastic volatility. One result restates a result of Coles *et al.* [7] within our framework, namely that effective risk aversion is greater in the presence of stochastic volatility than without. The other result provides comparative statics on the effect of marginal changes in the degree of stochastic volatility, something that is difficult to do without an explicit form for VARA. We show that absolute VARA is increasing in both constant absolute risk aversion and in the non-stochastic variance parameter. Next we examine the effects of changes in the distribution of absolute VARA, specifically a first- and second-order stochastic dominating shift in volatility. Finally, we finish this section with three applications illustrating absolute VARA with three commonly used volatility distributions.

In our analysis we make use of the fact that under exponential utility VARA can be expressed as the product of the constant absolute risk aversion γ and a function of the volatility distribution. To illustrate, we specify terminal wealth as $W = R_f + \omega(R-R_f)$, where R exhibits stochastic volatility as before. We can write expected utility $E[U(W)]$ as an iterated expectation $E[U(W)] = E_V\{E[U(W)|V]\}$ and by direct calculation we have

$$E[U(W)|V] = E[-\mathrm{e}^{-\gamma W}|V] = -\mathrm{e}^{-\gamma E[W|V] + \frac{1}{2}\gamma^2 \mathrm{Var}[W|V]}.$$

Since $E[W|V] = E[W] = (1-\omega)R_f + \omega E[R]$ and $\mathrm{Var}[W|V] = \omega^2\sigma^2 V$, we get that

$$E[U(W)|V] = -\mathrm{e}^{-\gamma\{(1-\omega)R_f + \omega E[R]\}}\mathrm{e}^{\frac{1}{2}\gamma^2\omega^2\sigma^2 V}.$$

By a similar argument as above,

$$E[U'(W)] = -\gamma \mathrm{e}^{-\gamma E[W]} E[\mathrm{e}^{\frac{1}{2}\gamma^2\omega^2\sigma^2 V}] \quad \text{and} \quad E^Q[U''(W)] = \gamma^2 \frac{\mathrm{e}^{-\gamma E[W]}}{E[V]} E[V\mathrm{e}^{\frac{1}{2}\gamma^2\omega^2\sigma^2 V}].$$

Using these results, we can now show that the absolute VARA is independent of the expected returns:

$$\Gamma^{\mathrm{CARA}} = \frac{-E^Q[U''(W)]}{E[U'(W)]} = \gamma \frac{E[V\mathrm{e}^{\frac{1}{2}\gamma^2\omega^2\sigma^2 V}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\omega^2\sigma^2 V}]E[V]}.$$

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For the representative investor where $\omega = 1$, we can simplify this expression to

$$\Gamma^{\text{CARA}} = \gamma \frac{E[V e^{\frac{1}{2}\gamma^2\sigma^2 V}]}{E[e^{\frac{1}{2}\gamma^2\sigma^2 V}]E[V]} = \gamma \frac{E^T[V]}{E[V]},$$

where the superscript 'T' indicates that expectations are taken under the exponentially tilted distribution $T(V) = e^{\frac{1}{2}\gamma^2\sigma^2 V} p(V)/E[e^{\frac{1}{2}\gamma^2\sigma^2 V}]$. This representation of absolute VARA through the tilted distribution facilitates the comparative static results that follow.

To provide insight into the role of the tilted distribution on portfolio choice, consider the representation of the optimal portfolio weight. Under constant absolute risk aversion we replace VARA, Γ, with the specialized absolute VARA, Γ^{CARA}, yielding

$$\omega^\star_{SV} = \frac{1}{\Gamma^{\text{CARA}}}\left(\frac{E[R]-R_f}{\text{Var}[R]}\right).$$

Using the definition of absolute VARA above, this simplifies to

$$\omega^\star_{SV} = \frac{1}{\gamma}\left(\frac{E[R]-R_f}{\text{Var}^T[R]}\right), \tag{7}$$

where $\text{Var}^T[R] = \sigma^2 E^T[V]$. These two equations highlight two ways of thinking about the effect of stochastic volatility on the optimal portfolio weight. In the top equation, one can think of investors using the expected return and variance of R and then behaving as if they have a risk aversion parameter that differs from their utility function risk aversion parameter γ. On the other hand, we can think of investors as having the utility function risk aversion parameter γ, but instead of using the variance of returns under stochastic variance to make decisions, $\text{Var}[R] = \sigma^2 E[V]$, investors apply the larger variance evaluated under the tilted distribution, $\text{Var}^T[R]$, as in the bottom equation. Thus the tilted distribution can be thought of as inducing the equivalent certain variance that leads the investor to the same portfolio allocation as under stochastic volatility. We now turn to our first set of results on the properties of absolute VARA.

Theorem 2
Absolute VARA Properties I

1. Absolute VARA is greater than the constant absolute risk aversion, that is, $\Gamma^{\text{CARA}} \geqslant \gamma$.
2. Absolute VARA, Γ^{CARA}, is increasing in constant absolute risk aversion γ and variance parameter, σ^2, that is

$$\frac{\partial \Gamma^{\text{CARA}}}{\partial \gamma} > 0, \quad \frac{\partial \Gamma^{\text{CARA}}}{\partial \sigma^2} > 0.$$

Proof
See Appendix B.

As noted above, the first result restates a prior finding within our framework. The second result, that an increase in the representative investor's constant risk aversion or in the non-stochastic variance increases the effective risk aversion, is new because it requires a measure of effective risk aversion such as the one we develop here. Next, we examine the effect of a first- and second-order stochastic dominating change in the volatility distribution on absolute VARA. We find that a first-order stochastic dominating shift in the volatility distribution does not necessarily increase absolute VARA, whereas a second-order stochastic dominating shift in the volatility distribution does increase absolute VARA.

Theorem 3
Absolute VARA Properties II

1. Consider two independent distributions with stochastic volatilities, V_1 and V_2, and associated absolute VARA, Γ_1^{CARA} and Γ_2^{CARA}. Let $V = V_1 + V_2$. Then $\Gamma^{\text{CARA}} = \pi\Gamma_1^{\text{CARA}} + (1-\pi)\Gamma_2^{\text{CARA}}$, where $\pi = E[V_1]/E[V_1+V_2]$.
2. Consider two distributions with stochastic volatilities, V and V_1, and associated absolute VARA, Γ^{CARA} and Γ_1^{CARA}. Suppose that $V = V_1 + Z$, where Z is independent of V_1, and $E[Z] = 0$, then $\Gamma^{\text{CARA}} \geqslant \Gamma_1^{\text{CARA}}$.

Proof
See Appendix C.

The first result appears surprising initially. One might expect that a first-order stochastic dominant shift in the distribution of volatility would result in a greater effective risk aversion. While a first-order stochastic dominating shift in the distribution

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of the variance increases the expected variance and the variance of the variance, absolute VARA is a ratio of the expectation of the second derivative of utility to the expectation of the marginal utility. The diversification effect of taking expectations over the sum of independent distributions partially offsets the effect of the greater mean and variance of the volatility distribution. Thus a first-order stochastic dominating shift in the variance distribution leaves the resulting absolute VARA in between the two variance components and absolute VARA is a sub-additive risk measure, since $V = V_1 + V_2$ implies that $\Gamma_1^{\mathrm{CARA}} + \Gamma_2^{\mathrm{CARA}} \geqslant \Gamma^{\mathrm{CARA}}$. Theorem 3 also demonstrates the effect of a second-order stochastic dominating shift in the distribution of volatility. An increase in the uncertainty about the variance leads to a higher absolute VARA. In this case there is no diversification effect to offset the greater variance of the volatility distribution.

The VARA representation allows us to explicitly examine how absolute VARA changes with particular assumptions. Consider the implications on absolute VARA of three volatility distributions of interest in financial economics. All the three are members of the generalized inverse Gaussian family: the exponential, the inverse gamma, and a special case of the generalized inverse Gaussian distribution.

First, suppose that V is distributed exponentially with density $p(V) = \lambda \mathrm{e}^{-\lambda V}$, a simple and parsimonious choice. The inverse of the parameter λ represents both the expected volatility and the standard deviation of volatility. Hence, the ratio of the expected volatility to the variance of volatility equals the constant λ. For sufficiently low risk aversion, $\gamma < \sqrt{2\lambda}/\sigma$, the absolute VARA is finite and can be computed as

$$\Gamma^{\mathrm{CARA}} = \frac{\gamma}{\left(\lambda - \frac{\gamma^2\sigma^2}{2}\right)}.$$

Note that $\partial\Gamma^{\mathrm{CARA}}/\partial\lambda < 0$. That is, an increase in the stochastic volatility scaled by the expected volatility increases absolute VARA. This is consistent with the effect of an increase in the known variance, σ^2, shown in Theorem 2. Combining this expression with our prior result on the optimal portfolio share for the risky asset, one can assess the effect of different volatility assumptions given measures of γ, risk premia, and volatility.

Not all distributions of asset returns result in a well-defined portfolio allocation problem, and volatility distributions associated with these distributions of asset returns will therefore have undefined VARA for every strictly positive constant absolute risk aversion γ. For example, the inverse gamma distribution with parameters $p(V) = \Gamma(\alpha)^{-1}\alpha^{\lambda} V^{\lambda-1}\mathrm{e}^{-\alpha/V}$ could be an attractive choice for the distribution of V. When volatility is distributed inverse gamma the distribution of returns is a t-distribution which arises naturally as the predictive distribution of returns for a Bayesian investor learning the return distribution parameters as in [11, 12, 21]. However, as these researchers and others have discussed, the expected utility of terminal wealth under exponential utility and t-distributed asset returns is undefined. Therefore the inverse gamma distribution for volatility cannot be used to evaluate absolute VARA and portfolio allocation.

Both of these distributions are part of a family of volatility distributions known as the Generalized Inverse Gaussian (GIG) family. Whenever volatility is distributed in the GIG family, the distribution of returns follows a hyperbolic distribution. Barndorff-Nielsen and Shephard [22] discuss additional financial econometric applications for this family of distributions. Another member of the GIG family that is more flexible than the exponential but still results in a well-defined absolute VARA is the case where $p(V)$ is given by

$$p(V|\alpha, \lambda) = I(\alpha, \lambda)^{-1} V^{-\frac{3}{2}} \mathrm{e}^{-\frac{1}{2}(\alpha V^{-1} + \lambda V)}$$

for a suitable normalization constant $I(\alpha, \lambda)$. Here the expected volatility is given by $E[V] = \sqrt{\alpha/\lambda}$ and the variance of volatility is $\mathrm{Var}[V] = \sqrt{\alpha/\lambda^3}$. For sufficiently low constant absolute risk aversion, $\gamma < \sqrt{\lambda}/\sigma$, absolute VARA is given by

$$\Gamma^{\mathrm{CARA}} = (\gamma^{-2} - \lambda^{-1}\sigma^2)^{\frac{-1}{2}}.$$

See Appendix D for detailed proofs. Note that Γ^{CARA} depends only on the market price of volatility $\lambda = E(V)/\mathrm{Var}(V)$ and the known volatility σ^2 as in the case of exponential volatility. In this case, it is straightforward to check their comparative statics that $\partial\Gamma^{\mathrm{CARA}}/\partial\lambda < 0$ and $\partial\Gamma^{\mathrm{CARA}}/\partial\alpha = 0$. Similar to the exponential distribution, absolute VARA is affected only by the constant ratio of the expected volatility to the variance of volatility λ.

4. Equilibrium prices under stochastic volatility

In this section we consider the equilibrium asset price implications when multiple investors choose optimal portfolios over one safe and multiple risky assets incorporating stochastic volatility effects. We endogenously derive the market clearing equilibrium stock prices when firm's future cash flows are affected by a single factor that exhibits stochastic volatility. The resulting equilibrium stock returns adhere to a CAPM relationship when there is stochastic volatility. While the fact

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that a CAPM relationship holds for stochastic volatility is well known‖ we provide something additional—an explicit mean-variance result in terms of volatility-adjusted risk aversion with endogenous equilibrium prices.

Consider I investor types indexed by $i=1,\ldots,I$ and J firms indexed by $j=1,\ldots,J$. Investors differ in their initial wealth, W_i^0, their terminal wealth, W_i, and their utility functions over terminal wealth, $U_i(W_i)$. The terminal value of each firm j's stock is a random variable, X_j, with three elements: a firm-specific mean, μ_j, market-wide random factor, F, and a firm specific, idiosyncratic shock, ε_j. Thus we write

$$X_j=\mu_j+b_jF+\varepsilon_j,$$

where b_j is the sensitivity of firm j's terminal value to the market-wide factor and $\varepsilon_j\sim N(0,\sigma_\varepsilon^2)$. Stochastic volatility enters the problem in the distribution of the market factor, $p(F)$, with

$$p(F|V)\sim N(0,V\sigma_F^2) \quad \text{and} \quad V\sim p(V).$$

The conditional distribution of the value of stock j is then

$$p(X_j|V)\sim N(\mu_j,Vb_j^2\sigma_F^2+\sigma_\varepsilon^2).$$

Given that all parameters and distributions are common knowledge, each investor maximizes expected utility of terminal wealth subject to the budget constraint by choosing the dollars to invest in bonds, B_i, and the dollars to invest in firm j. We write the dollars invested in firm j as $\omega_{i,j}P_j$, where $\omega_{i,j}$ the fraction of the firm that the investor purchases at a price of P_j. Thus each investor i solves

$$\max_{B_i,\omega_{i,j}} E[U_i(W_i)] \quad \text{subject to } W_0^i \geqslant B_i+\sum_j \omega_{i,j}P_j$$

and final wealth is equal to end of period value of the investment,

$$W_i=B_iR_f+\sum_j \omega_{i,j}X_j.$$

Our main result in this section is that the market clearing equilibrium prices under stochastic volatility follow an explicit CAPM relationship as described below.

Theorem 4 (*Equilibrium prices under stochastic volatility*)
Let $W_m=\sum_i W_i$ be aggregate final wealth, $P_m=\sum_j P_j$ be the initial value of all stocks, and $X_m=\sum_j X_j$ be the end of period aggregate value of all stocks. Then the initial equilibrium price of stock j for individual investors with end of period utility $U_i(W_i)$ is

$$P_j=R_f^{-1}(E[X_j]-\Gamma_m\mathrm{Cov}[W_m,X_j]) \ \forall j,$$

where $\Gamma_m=(\sum_i \Gamma_i^{-1})^{-1}$ is the aggregate volatility-adjusted risk aversion and Γ_i is the VARA of investor i. Furthermore, a CAPM pricing equation holds

$$E[R_j]-R_f=\beta_j(E[R_m]-R_f),$$

where the beta, $\beta_j=\mathrm{Cov}[R_j,R_m]/\mathrm{Var}[R_m]$ is the risk premium.

Proof
See Appendix E.

The first part of Theorem 4 establishes that volatility-adjusted risk aversion plays a role in determining risk premia in equilibrium prices. In this case investors' risk aversions toward stochastic volatility is aggregated into prices in an intuitive manner. The risk associated with stochastic volatility that is priced derives from how well a stock provides a hedge against movements in the economy-wide wealth. This is captured by the covariance between the future payoff from the stock and the future economy-wide wealth. Further, the effect of individual investors' risk preferences regarding stochastic volatility materializes in equilibrium prices only through the appropriately aggregated VARA, Γ_m. This aggregation of risk preferences has an analog in Wilson's [26] characterization of Pareto optimal risk sharing.

The second part of Theorem 4 establishes the role of stochastic volatility in equilibrium returns. Compared to a setting with certain variance, the presence of stochastic volatility affects the distribution of returns on individual stocks and returns

‖*Chamberlain* [17] *and Owen and Rabinovitch* [23] *show that elliptical distributions, and hence stochastic volatility distributions, lead to a CAPM. For CAPM results without stochastic volatility, see Ingersoll Jr* [24], *Huang and Litzenberger* [5], *and Cochrane* [25].

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on the market portfolio, which in turn affects the CAPM beta. In addition, the expected excess return on individual stocks and on the market portfolio are affected by stochastic volatility through equilibrium prices. In contrast to the effect on equilibrium prices, however, the calculation of the systematic risk factor can be undertaken without knowledge of individual investors' risk preferences.

In a sense the calculation of the beta, namely $\beta_j = \mathrm{Cov}[R_j, R_m]/\mathrm{Var}[R_m]$, is standard. It is affected by stochastic volatility solely from the fact that the distribution of returns depends on $p(V)$ via $p(R_j) = \int_0^\infty p(R_j|V)p(V)\,\mathrm{d}V$. However, given prices, there is no effect from size-biasing $Q(V)$ or from tilting the volatility distribution $T(V)$. This is in contrast to the calculation of CAPM betas when investors face asymmetric parameter uncertainty that does not materialize in realized returns. In that case, Coles and Loewenstein [27] and Coles *et al.* [7] show that an adjustment is required in the calculation of the betas.

Our final result reconsiders the case where all investors have constant absolute risk aversion and so stochastic volatility appears through absolute VARA. We show that in this case stochastic volatility affects the risk premium in prices only through the variance calculated under the tilted distribution.

Theorem 5 (*Equilibrium prices under CARA*)
Consider a representative investor with constant absolute risk aversion, $\gamma_m = (\sum_i \gamma_i^{-1})^{-1}$ in the setting of Theorem 4 who chooses between one risky and one safe asset for a single period investment horizon. Then the equilibrium price of the stock is given by

$$P = R_f^{-1}(E[X] - \Gamma_m^{\mathrm{CARA}}\mathrm{Var}[X]) = R_f^{-1}(\mu - \gamma_m \mathrm{Var}^T[X]),$$

where 'T' is tilted distribution for γ_m defined as before**.

Proof
See Appendix F.

As expected, the equilibrium price separates into a mean effect from the discounted expected future payouts and a variance effect due to investor's risk aversion. Theorem 5 expresses the variance effect in two ways. First, the relationship between prices and variances is characterized by the aggregate absolute VARA, Γ_m^{CARA}. Alternatively, the risk premium in prices can be conveniently expressed from the investors' aggregate risk aversion, γ_m, by using the variance calculated under the tilted distribution, instead of the variance calculated under the true probability measure. The appropriate tilted distribution uses the constant absolute risk aversion of the representative investor, γ_m. While each investor's absolute VARA can be expressed based on the tilted probability measure, we have identified a change in probability measure that converts an economy with stochastic volatility into an equivalent economy without stochastic volatility. This change of measure is a natural analog to the standard change to the equivalent martingale measure, which converts prices arising in an economy with risk-averse investors into the prices that would have prevailed in an equivalent economy with risk neutral investors. The application of the latter, standard change of probability measure is apparent after application of the tilted probability measure in this paper. As we rely on the market-clearing condition, our construction of the titled measure is unique. In contrast, the construction of the martingale measure from no arbitrage arguments need not be unique with an incomplete stock market whose securities' payoffs do not span all Arrow–Debreu securities.

5. Conclusions

In this paper we explore the effects of stochastic volatility on individual investment behavior, equilibrium prices, and market portfolios. We propose VARA as the appropriate measure of effective risk aversion when investors face asset returns with stochastic volatility. To establish that investors are effectively more risk averse when subject to stochastic volatility we summarize our results in the special case without wealth effects. We find that investors with exponential utility prefer constant variance of a given magnitude to stochastic volatility with expected variance equal to the constant variance. We also find that investor's trade-off increases in the level of variance for reductions in the variance of the volatility distribution. Further, we show that results such as the optimal share of wealth to invest in a risky asset and CAPM have analogous relationships in the stochastic volatility setting, but the parameters must be adjusted to take account of stochastic volatility.

In order to develop our analysis, we extend Stein's lemma to the case of normal distributions with stochastic volatility. This allows us to separate utility and covariance effects and provides tractable, interpretable results. While we derive 'Stein-like' relationships for distributions with stochastic volatility, it is important to note that the original Stein's lemma does not hold

**This suffices when two fund separation holds and investors behave as if they purchased a mix of the risk-free bond and a single stock portfolio.*

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in these situations. In each case, the new identity includes a different proportionality factor which has important behavioral implications. For example, we show that with particular objective functions, stochastic volatility causes decision makers to act more risk averse provided that the risk aversion is appropriately adjusted through the change of measure. Therefore, when uncertainty in the form of stochastic volatility changes over time, observed behavior should change as well.

Our work is related to the literature that examines the effect of estimation risk on portfolio choice. Our analysis primarily examines the effect of estimation risk in the volatility while existing research has also evaluated the effect of estimation risk in the mean. This leads to a number of avenues for future research, for example, combining the effects of estimation risk in both the mean and the volatility; extending the results to a multivariate stochastic volatility setting using the techniques derived in [28]; and finally considering inter-temporal portfolio problems with stochastic volatility along the lines of Balvers and Mitchell [29].

Appendix A: Proof of Theorem 1

First consider $\mathrm{Cov}[g(X), X]$. Using the Law of Iterated Expectations

$$\mathrm{Cov}[g(X), X] = E[g(X)(X - E[X])] = E_V\{E_{X|V}[g(X)(X - E[X])]\}$$
$$= E_V\{E_{X|V}[g(X)(X - E[X|V])]\}.$$

The result now follows from applying Stein's lemma conditionally as $X|V$ is normally distributed as $N(\mu, \sigma^2 V)$. Hence we have

$$\mathrm{Cov}[g(X), X] = E_V\{E_{X|V}[g^{'}(X)]\mathrm{Var}[X|V]\}.$$

Now $\mathrm{Var}[X|V] = \sigma^2 V$ and so

$$\mathrm{Cov}[g(X), X] = E_V\{E_{X|V}[g^{'}(X)]\sigma^2 V\} = E\left\{g^{'}(X)\frac{V}{E[V]}\right\}\sigma^2 E[V],$$

which implies that

$$\mathrm{Cov}[g(X), X] = E^Q[g^{'}(X)]\sigma^2 E[V]. \tag{A1}$$

Second, consider $\mathrm{Cov}[g(X), Y]$. In the bivariate case, we can write $Y = a + bX + \varepsilon$, where $\mathrm{Cov}[X, \varepsilon] = 0$ and $b = [Y, X]/\mathrm{Var}[X]$. Note that $\mathrm{Cov}[g(X), \varepsilon] = 0$ by construction. Moreover, we can write $\mathrm{Var}[X] = E_V\{\mathrm{Var}[X|V]\} + \mathrm{Var}_V\{E[X|V]\}$ where $\mathrm{Var}[X|V] = \sigma^2 V$ and $\mathrm{Var}_V\{E[X|V]\} = 0$. Hence $\mathrm{Var}[X] = \sigma^2 E[V]$. Hence from the univariate version of Stein's Lemma above we have

$$\mathrm{Cov}[g(X), Y] = E^Q[g^{'}(X)]\mathrm{Cov}[X, Y].$$

Appendix B: Absolute VARA properties

For these proofs it is useful to use the following simplified presentation of Γ^{CARA}. Let $T(V) = (e^{\frac{1}{2}\gamma^2\sigma^2 V}/E[e^{\frac{1}{2}\gamma^2\sigma^2 V}])p(V)$ and let E^T denote the expectation under this exponentially tilted volatility distribution. Then Γ^{CARA} can be represented as

$$\Gamma^{\mathrm{CARA}} = \gamma\frac{E^T[V]}{E[V]} = \gamma\frac{E[Ve^{\frac{1}{2}\sigma^2\gamma^2 V}]}{E[e^{\frac{1}{2}\sigma^2\gamma^2 V}]E[V]}.$$

As expected this is clearly unaffected by expected wealth, μ due to the choice of exponential utility.

This can be decomposed further by using the definition of covariance and writing

$$E[Ve^{\frac{1}{2}\gamma^2\sigma^2 V}] = E[V]E[e^{\frac{1}{2}\gamma^2\sigma^2 V}] + \mathrm{Cov}[V, e^{\frac{1}{2}\gamma^2\sigma^2 V}].$$

Hence

$$\Gamma^{\mathrm{CARA}} = \gamma + \gamma\frac{\mathrm{Cov}[V, e^{\frac{1}{2}\gamma^2\sigma^2 V}]}{E[e^{\frac{1}{2}\gamma^2\sigma^2 V}]E[V]}.$$

The first term is the coefficient of absolute risk aversion whereas the second term is positive and $\Gamma^{\mathrm{CARA}} \geqslant \gamma$. To see this note that since $e^{\frac{1}{2}\gamma^2\sigma^2 V}$ is increasing in the variance, V, its covariance with the variance is also non-negative.

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Appendix C: Proof of VARA dominance results

Proof of Theorem 2
By differentiation, we find that

$$\Gamma^{\text{CARA}} = \gamma \frac{E[V \mathrm{e}^{\frac{1}{2}\sigma^2\gamma^2 V}]}{E[\mathrm{e}^{\frac{1}{2}\sigma^2\gamma^2 V}]E[V]},$$

$$\frac{\partial \Gamma^{\text{CARA}}}{\partial \sigma^2} = \frac{\gamma^2}{2E[V]}\left\{ \frac{E[V^2 \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]} - \left(\frac{E[V \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]}\right)^2\right\}.$$

Recognizing the exponential tilt distribution $T(V) = \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V} p(V)/E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]$ we obtain

$$\frac{\partial \Gamma^{\text{CARA}}}{\partial \sigma^2} = \frac{\gamma^2}{2E[V]}\{E^T[V^2] - (E^T[V])^2\} = \frac{\gamma^2}{2E[V]}\{\text{Var}^T[V]\}.$$

Hence $\partial\Gamma^{\text{CARA}}/\partial\sigma^2 \geqslant 0$ as claimed.

Similarly, differentiation with respect to γ yields

$$\frac{\partial \Gamma^{\text{CARA}}}{\partial \gamma} = \frac{\Gamma^{\text{CARA}}}{\gamma} + \frac{\gamma\sigma^2}{E[V]}\left\{ \frac{E[V^2 \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]} - \left(\frac{E[V \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]}\right)^2\right\} = \frac{\Gamma^{\text{CARA}}}{\gamma} + \frac{\gamma\sigma^2}{E[V]}\text{Var}^T[V] \geqslant 0.$$

Proof of Theorem 3
Consider Γ^{CARA} for the volatility distribution $V = V_1 + V_2$, where V_1 and V_2 are independent. Then

$$\Gamma^{\text{CARA}} = \gamma \frac{E[V \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}]E[V]} = \gamma \frac{E[(V_1+V_2)\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2(V_1+V_2)}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2(V_1+V_2)}]E[V]} = \frac{\gamma}{E[V]}\left\{ \frac{E[V_1 \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V_1}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V_1}]} + \frac{E[V_2 \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V_2}]}{E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V_2}]}\right\}.$$

Define the tilted distributions $T_1(V_1) = \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V_1} p(V_1)/E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V_1}]$ and $T_2(V_2) = \mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V_2} p(V_2)/E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V_2}]$. Hence

$$\Gamma^{\text{CARA}} = \frac{\gamma}{E[V]}\{E^{T_1}[V_1] + E^{T_2}[V_2]\} = \frac{1}{E[V]}\{E[V_1]\Gamma_1^{\text{CARA}} + E[V_2]\Gamma_1^{\text{CARA}}\}.$$

Defining $\pi = E[V_1]/E[V_1+V_2]$ the first result now follows:

$$\Gamma^{\text{CARA}} = \pi\Gamma_1^{\text{CARA}} + (1-\pi)\Gamma_2^{\text{CARA}}.$$

A similar argument holds if we define $Z = V_2$, where $E[Z] = 0$ and $E[V] = E[V_1]$. In that case, $E^{T_2}[V_2] \geqslant 0$ and $\Gamma^{\text{CARA}} \geqslant \Gamma_1^{\text{CARA}}$ as claimed.

Appendix D: Proof for generalized inverse Gaussian-distributed volatility

Consider the Generalized Inverse Gaussian (GIG) distribution defined by

$$p(V|\alpha,\lambda) = \frac{V^{-\frac{3}{2}}}{I(\alpha,\lambda)} \mathrm{e}^{-\frac{1}{2}(\alpha V^{-1} + \lambda V)},$$

where the normalization constant, $I(\alpha,\lambda)$, ensures that $p(V)$ is a density. We will prove the results stated in the paper through a sequence of lemmas. In our proofs, the following identity from Andrews and Mallows [19, p. 100] will be useful

$$\int_0^\infty \mathrm{e}^{-\frac{1}{2}(a^2u^2 + b^2u^{-2})}\,\mathrm{d}u = \left(\frac{\pi}{2a^2}\right)^{\frac{1}{2}} \mathrm{e}^{-|ab|}. \tag{D1}$$

Lemma D1
The normalization constant $I(\alpha,\lambda) = \sqrt{2\pi/\alpha}\,\mathrm{e}^{-\sqrt{\alpha\lambda}}$ and for $0 < \gamma < \sqrt{\lambda}/\sigma$

$$E[\mathrm{e}^{\frac{1}{2}\gamma^2\sigma^2 V}] = \mathrm{e}^{-\sqrt{\alpha(\lambda-\gamma^2\sigma^2)}}\mathrm{e}^{-\sqrt{\alpha\lambda}}. \tag{D2}$$

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Proof of Lemma D1

$$E\left[e^{\frac{1}{2}\gamma^2\sigma^2 V}\right] = \int_0^\infty e^{-\frac{1}{2}\gamma^2\sigma^2 V} p(V)\mathrm{d}V = \int_0^\infty e^{-\frac{1}{2}\gamma^2\sigma^2 V} \frac{V^{-\frac{3}{2}}}{I(\alpha,\lambda)} e^{\frac{1}{2}(\alpha V^{-1}+\lambda V)}\mathrm{d}V$$
$$= \frac{-2}{I(\alpha,\lambda)}\int_0^\infty e^{-\frac{1}{2}(\alpha V^{-1}+(\lambda-\gamma^2\sigma^2)V)}(-\frac{1}{2}V^{-\frac{3}{2}})\mathrm{d}V,$$

where $I(\alpha,\lambda)$ is a normalization constant. Changing of variables $u=V^{-\frac{1}{2}}$ yields

$$E[e^{\frac{1}{2}\gamma^2\sigma^2 V}] = \frac{2}{I(\alpha,\lambda)}\int_0^\infty e^{-\frac{1}{2}(\alpha u^2+(\lambda-\gamma^2\sigma^2)u^{-2})}\mathrm{d}u.$$

Provided that $(\lambda-\gamma^2\sigma^2)\geqslant 0$, we can apply (D1) and find that

$$E[e^{\frac{1}{2}\gamma^2\sigma^2 V}] = \frac{2}{I(\alpha,\lambda)}\sqrt{\frac{\pi}{2\alpha}}e^{-\sqrt{\alpha(\lambda-\gamma^2\sigma^2)}}. \tag{D3}$$

In the special case where $\gamma=0$ it follows that

$$1 = \frac{2}{I(\alpha,\lambda)}\sqrt{\frac{\pi}{2\alpha}}e^{-\sqrt{\alpha\lambda}}$$

or

$$I(\alpha,\lambda) = \int_0^\infty V^{-\frac{3}{2}} e^{\frac{1}{2}(\alpha V^{-1}+\lambda V)}\mathrm{d}V = \sqrt{\frac{2\pi}{\alpha}}e^{-\sqrt{\alpha\lambda}}.$$

The second part of the lemma now follows from substitution back into (D3).

Lemma D2
For any constant γ such that $0<\gamma<\sqrt{\lambda}/\sigma$,

$$E[Ve^{\frac{1}{2}\gamma^2\sigma^2 V}] = \sqrt{\frac{\alpha}{(\lambda-\gamma^2\sigma^2)}}e^{-\sqrt{\alpha(\lambda-\gamma^2\sigma^2)}}e^{-\sqrt{\alpha\lambda}}. \tag{D4}$$

Proof of Lemma D2

$$E[Ve^{\frac{1}{2}\gamma^2\sigma^2 V}] = \int_0^\infty Ve^{\frac{1}{2}\gamma^2\sigma^2 V} p(V)\mathrm{d}V = \int_0^\infty Ve^{\frac{1}{2}\gamma^2\sigma^2 V}\frac{V^{-\frac{3}{2}}}{I(\alpha,\beta)}e^{-\frac{1}{2}(\alpha V^{-1}+\lambda V)}\mathrm{d}V$$
$$= \frac{2}{I(\alpha,\lambda)}\int_0^\infty e^{-\frac{1}{2}(\alpha V^{-1}+(\lambda-\gamma^2\sigma^2)V)}(\frac{1}{2}V^{-\frac{1}{2}})\mathrm{d}V.$$

Again change of variables $u=V^{-1/2}$ yields

$$E[Ve^{\frac{1}{2}\gamma^2\sigma^2 V}] = \frac{2}{I(\alpha,\lambda)}\int_0^\infty e^{-\frac{1}{2}(\alpha u^{-2}+(\lambda-\gamma^2\sigma^2)u^2)}\mathrm{d}u.$$

Assuming that $(\lambda-\gamma^2\sigma^2)\geqslant 0$, it follows from (D1) that

$$E[Ve^{\frac{1}{2}\gamma^2\sigma^2 V}] = \frac{2}{I(\alpha,\lambda)}\sqrt{\frac{\pi}{2(\lambda-\gamma^2\sigma^2)}}e^{-\sqrt{\alpha(\lambda-\gamma^2\sigma^2)}}.$$

Substituting for $I(\alpha,\lambda)$ yields (D4) immediately.

Lemma D3
The expected and variance of stochastic volatility are $E[V]=\sqrt{\alpha/\lambda}$, $\mathrm{Var}[V]=\sqrt{\alpha/\lambda^3}$.

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Proof of Lemma D3
First note that $E[V]$ arises as the special case of Lemma D2, where $\lambda=0$. Second note that

$$E[V^2]=\int_0^\infty V^2 p(V)\,dV=\int_0^\infty V^2\frac{V^{-\frac{3}{2}}}{I(\alpha,\lambda)}e^{-\frac{1}{2}(\alpha V^{-1}+\lambda V)}\,dV=\frac{1}{I(\alpha,\lambda)}\int_0^\infty\left\{V^{\frac{1}{2}}e^{-\frac{\alpha}{2}V^{-1}}\right\}e^{-\frac{\lambda}{2}V}\,dV.$$

Integration by parts gives

$$E[V^2]=\frac{1}{\lambda I(\alpha,\lambda)}\left\{\int_0^\infty V^{\frac{-1}{2}}e^{-\frac{1}{2}(\alpha V^{-1}+\lambda V)}\,dV+\alpha\int_0^\infty V^{\frac{-3}{2}}e^{-\frac{1}{2}(\alpha V^{-1}+\lambda V)}\,dV\right\}.$$

We now apply Lemma D2 with $\gamma=0$ and find

$$E[V^2]=\frac{1}{\lambda}\{E[V]+\alpha\}=\sqrt{\frac{\alpha}{\lambda^3}}+\frac{\alpha}{\lambda},$$

where the last equality follows from $E[V]$. The result follows from the definition of variance

$$\mathrm{Var}[V]=E[V^2]-(E[V])^2=\sqrt{\frac{\alpha}{\lambda^3}}+\frac{\alpha}{\lambda}-\left(\sqrt{\frac{\alpha}{\lambda}}\right)^2=\sqrt{\frac{\alpha}{\lambda^3}}.$$

Lemma D4
Consider a representative investor with constant absolute risk aversion, γ, that is below the constant $\sqrt{\lambda}/\sigma$. If V is distributed with the GIG distribution specified above, then VARA becomes

$$\Gamma^{\mathrm{CARA}}=\left(\gamma^{-2}-\frac{\sigma^2}{\lambda}\right)^{-\frac{1}{2}}.$$

Proof of Lemma D4
From Section 3, the volatility-adjusted risk aversion can be expressed as

$$\Gamma^{\mathrm{CARA}}=\gamma\frac{E[Ve^{\frac{1}{2}\gamma^2\sigma^2V}]}{E[e^{\frac{1}{2}\gamma^2\sigma^2V}]E[V]}.$$

Using the previous lemmas, we calculate the terms in Γ^{CARA} as follows. Since

$$\frac{E[Ve^{\frac{1}{2}\gamma^2\sigma^2V}]}{E[e^{\frac{1}{2}\gamma^2\sigma^2V}]}=\sqrt{\frac{\alpha}{(\lambda-\gamma^2\sigma^2)}}.$$

It follows that

$$\Gamma^{\mathrm{CARA}}=\gamma\frac{E[Ve^{\frac{1}{2}\gamma^2\sigma^2V}]}{E[e^{\frac{1}{2}\gamma^2\sigma^2V}]E[V]}=\gamma\sqrt{\frac{\alpha}{(\lambda-\gamma^2\sigma^2)}}\sqrt{\frac{\lambda}{\alpha}}=\left(\gamma^{-2}-\frac{\sigma^2}{\lambda}\right)^{-\frac{1}{2}}$$

as required. The comparative statics follow immediately.

Appendix E: Proof of Theorem 4

We first derive the market clearing prices and then prove that CAPM holds.

Equilibrium prices

Assuming that the budget constraint is binding, terminal wealth is given by

$$W_i=\left(W_0^i-\sum_j\omega_{i,j}P_j\right)R_f+\sum_j\omega_{i,j}X_j=W_0^iR_f+\sum_j\omega_{i,j}(X_j-P_jR_f).$$

Hence, each investor chooses $\{\omega_{i,j}|j=1,\ldots,J\}$ to maximize expected utility $E[U_i(W_i)]$. The first-order conditions can then be written as

$$0=E[U_i'(W_i)(X_j-P_jR_f)]\forall i\,\forall j.$$

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Proceeding as in section 2 we use the definition of covariance for the first-order condition

$$\mathrm{Cov}[U_i'(W_i), X_j] = E[U_i'(W_i)]E[(X_j - P_j R_f)] \,\forall i \,\forall j.$$

From Stein's lemma with stochastic volatility, we get

$$\mathrm{Cov}[U_i'(W_i), X_j] = E^Q[U_i''(W_i)]\mathrm{Cov}[W_i, X_j].$$

Substituting this into the above equation yields

$$E[U_i'(W_i)]E[(X_j - P_j R_f)] = E^Q[U_i''(W_i)]\mathrm{Cov}[W_i, X_j] \,\forall i, j$$

or

$$\Gamma_i^{-1} E[(X_j - P_j R_f)] = \mathrm{Cov}[W_i, X_j] \,\forall i, j \tag{E1}$$

where

$$\Gamma_i = \frac{-E^Q[U_i''(W_i)]}{E[U_i'(W_i)]}$$

is the absolute VARA of investor i.

Since U_i is strictly increasing and concave, taking expectations under the size-biased measure preserves that $\Gamma_i > 0$. Summing on both sides of (E1) over all investors in the market

$$\sum_i \Gamma_i^{-1} E[(X_j - P_j R_f)] = \sum_i \mathrm{Cov}[W_i, X_j] \,\forall j$$

and using the linearity of the covariance, we get

$$\Gamma_m E[(X_j - P_j R_f)] = \mathrm{Cov}[(\sum_i W_i), X_j] \,\forall j,$$

where $\Gamma_m = (\sum_i \Gamma_i^{-1})^{-1}$ is the volatility-adjusted aggregate absolute risk aversion of the economy.

The relation to economy-wide terminal wealth, W_m, economy-wide initial wealth, W_m^0, and the risky return on the market portfolio, R_m, can be summarized as

$$W_m = \left(\sum_i W_i\right) = \left(\sum_i B_i\right) R_f + \left(W_m^0 - \sum_i B_i\right) R_m,$$

where $R_m = X_m / (\sum_j P_j)$ and $X_m = \sum_j X_j$.

By substitution, the excess return on stock j can be written as

$$E[(X_j - P_j R_f)] = \Gamma \mathrm{Cov}[R_m, X_j] \quad \forall j, \tag{E2}$$

where

$$\Gamma = \Gamma_m \left(W_m^0 - \sum_i B_i\right)$$

measures the volatility-adjusted aggregate relative risk aversion of the economy. Alternatively, the price of stock j can be written as

$$P_j = R_f^{-1}(E[X_j] - \Gamma_m^{-1}\mathrm{Cov}[W_m, X_j]) \,\forall j. \tag{E3}$$

Capital asset pricing model (*CAPM*)

Dividing (E2) by the initial market value of stock, P_j, we find that

$$E[(R_j - R_f)] = \Gamma_m \mathrm{Cov}[R_m, R_j] \,\forall j, \tag{E4}$$

where $R_j = (X_j / P_j)$ is the return of stock j. Multiplying both sides by the fraction of stock j in the market portfolio, $\omega_{m,j} = P_j / \sum_k P_k = P_j / (W_m^0 - \sum_i B_i)$,

$$E[\omega_{m,j}(X_j - P_j R_f)] = \Gamma_m \mathrm{Cov}[X_m, \omega_{m,j} X_j] \quad \forall j$$

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and summing (E4) over all stocks

$$E[\sum_j \omega_{m,j}(X_j - P_j R_f)] = \Gamma_m \text{Cov}[X_m, \sum_j \omega_{m,j} X_j]$$

yields

$$E[X_m] - (\sum_j P_j) R_f = \Gamma_m \text{Var}[X_m].$$

Dividing by the initial investment in the market portfolio yields

$$E[R_m] - R_f = \Gamma_m (\sum_j P_j) \text{Var}[R_m],$$

where, as before, R_m is the return on the market portfolio. This reduces to

$$\Gamma_m (\sum_j P_j) = \frac{E[R_m] - R_f}{\text{Var}[R_m]}.$$

Hence, the presence of stochastic volatility does not change the basic insight that in the aggregate the risk premium for market risk is strictly positive. By substitution of Γ_m the excess return on stock j can be written as

$$E[(R_j - R_f)] = (\frac{E[R_m] - R_f}{\text{Var}[R_m]}) \text{Cov}[R_m, R_j] \ \forall j,$$

the familiar CAPM form.

Appendix F: Proof of Theorem 5

Each investor with exponential utility buys the fraction ω_i of the stock and invests the remainder in the safe bond. Formally, investor i solves

$$\max_{\omega_i} E[U(W_i)] = -\gamma_i^{-1} \mathrm{e}^{-\gamma_i W_0^i R_f} E[\mathrm{e}^{-\gamma_i \omega_i (X - P R_f)}] \ \forall i$$

or since $X|V \sim N(\mu, \sigma^2 V)$

$$\min_{\omega_i} -\gamma_i(\omega_i(\mu - P R_f)) + \ln(E_V[\mathrm{e}^{\frac{1}{2}\gamma_i^2 \omega_i^2 \sigma^2 V}]) \ \forall i.$$

The first-order conditions are

$$0 = -\gamma_i(\mu - P R_f) + \gamma_i^2 \omega_i \sigma^2 \frac{E_V[V \mathrm{e}^{\frac{1}{2}\gamma_i^2 \omega_i^2 \sigma^2 V}]}{E_V[\mathrm{e}^{\frac{1}{2}\gamma_i^2 \omega_i^2 \sigma^2 V}]} \ \forall i$$

and the associated second-order conditions for ω_i are

$$\gamma_i^2 \sigma^2 \frac{E_V[V \mathrm{e}^{\frac{1}{2}\gamma_i^2 \omega_i^2 \sigma^2 V}]}{E_V[\mathrm{e}^{\frac{1}{2}\gamma_i^2 \omega_i^2 \sigma^2 V}]} + \gamma_i^2 \omega_i^2 \sigma^2 \text{Var}^{T_i}[V] \geqslant 0 \ \forall \gamma_i \ \forall i.$$

Rearranging terms, we find that each investor's implicit demand function for stock is

$$\omega_i(P) = \frac{1}{\Gamma_i^{\text{CARA}}} \left(\frac{\mu - P R_f}{E[V]\sigma^2} \right) \quad \forall i, \tag{F1}$$

where

$$\Gamma_i^{\text{CARA}} = \gamma_i \frac{E_V[V \mathrm{e}^{\frac{1}{2}\gamma_i^2 \omega_i^2 \sigma^2 V}]}{E_V[\mathrm{e}^{\frac{1}{2}\gamma_i^2 \omega_i^2 \sigma^2 V}] E[V]}$$

is the absolute VARA defined in Section 3. Proceeding as in the proof of Theorem 4, we sum across all investors in (F1), define $\Gamma_m^{\text{CARA}} = (\sum_i (\Gamma_i^{\text{CARA}})^{-1})^{-1}$, and apply the market clearing condition that $\sum_i \omega_i(P) = 1$. This yields the equilibrium price:

$$P^* = R_f^{-1}(\mu - \Gamma_m^{\text{CARA}} E[V]).$$

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To complete the proof we conjecture and verify that the optimal portfolio allocation is independent of the degree of stochastic volatility and equals $\omega_i(P^*) = \gamma_i^{-1}\gamma_m$, where $\gamma_m = (\sum_i \gamma_i^{-1})^{-1}$ ensures market clearing. Since

$$\Gamma_i^{\text{CARA}} = \gamma_i \frac{E_V[V e^{\frac{1}{2}\gamma_m^2 \sigma^2 V}]}{E_V[e^{\frac{1}{2}\gamma_m^2 \sigma^2 V}]E[V]} \forall i$$

it follows that

$$\Gamma_m^{\text{CARA}} = \gamma_m \frac{E_V[V e^{\frac{1}{2}\gamma_m^2 \sigma^2 V}]}{E_V[e^{\frac{1}{2}\gamma_m^2 \sigma^2 V}]E[V]}.$$

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