

# The Impact of Jumps in Volatility and Returns

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## Abstract

This paper examines a class of continuous-time models with stochastic volatility that incorporate jumps in returns and volatility. We develop a likelihood-based estimation strategy and provide estimates of model parameters, spot volatility, jump times and jump sizes using S&P 500 and Nasdaq 100 index returns. Estimates of jump times, jump sizes and volatility are particularly useful for identifying the effects of these factors during periods of market stress, such as those in 1987, 1997 and 1998. Using both formal and informal diagnostics, we find strong evidence for jumps in volatility, even after accounting for jumps in returns. Finally, we study the impact of these factors and of estimation risk on option prices.

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A surprising recent finding indicates that models with both diffusive stochastic volatility and jumps in returns are incapable of fully capturing the empirical features of equity index returns or option prices.<sup>1</sup> Empirical evidence indicates that the conditional volatility of returns rapidly increases, a movement difficult to generate using a diffusion specification for volatility and jumps in returns.<sup>2</sup> In this paper, we propose to remedy this problem by incorporating jumps in stochastic volatility, and we provide empirical evidence supporting the presence and importance of this additional factor.

Intuitively, what is it that jumps in volatility provide that jumps in returns and diffusive stochastic volatility cannot? Jumps in returns can generate large movements such as the crash of 1987, but the impact of a jump is transient: a jump in returns today has no impact on the future distribution of returns. On the other hand, diffusive volatility is highly persistent, but its dynamics are driven by a Brownian motion. For this reason, diffusive stochastic volatility can only increase gradually via a sequence of small normally distributed increments. Jumps in volatility fill the gap between jumps in returns and diffusive volatility by providing a rapidly moving but persistent factor that drives the conditional volatility of returns.

We focus on the role of jumps in volatility and returns in S&P 500 and Nasdaq 100 index returns, two prominent indices with actively traded futures and European option contracts. Our empirical approach departs from the usual routine of estimating parameters and performing specification tests in that we also estimate the unobserved jump times, jump sizes and spot volatilities. These estimates provide a dynamic picture of the roles these factors play and are useful for analyzing periods of market stress. It is especially important to determine the contribution of jumps to periods of market stress because jump risk, either

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<sup>1</sup>See, for example, Bakshi et al. (1997), Bates (2000) and Pan (2002).

<sup>2</sup>Bates (2000), Duffie, et al. (2000) and Pan (2002) provide evidence for the presence of positive jumps in volatility. For example, Bates (2000, p. 219) argues, referring to large increases observed in volatility, that “the high improbability of those outliers given the diffusion assumptions indicates that the true conditional transition distribution is far more leptokurtotic than hypothesized, and suggests that the factors underlying option prices follow jump processes.”

in returns or in volatility, cannot typically be hedged away, and investors may demand a large premia to carry these risks.<sup>3</sup>

We consider two models with jumps in volatility and returns, one with contemporaneous arrivals and correlated jump sizes and another with independent arrivals and sizes, both introduced by Duffie, et al. (2000). In these models, we find strong evidence for the presence of both jumps in volatility and returns. First, adding jumps in returns to the square-root stochastic volatility model dramatically changes the behavior of stochastic volatility. At certain points in time, the addition of jumps reduces annualized spot volatility by as much as 20%, from, 50% to 30%. While jumps in returns are infrequent events (1-2 per year), they are typically large and explain 8-15% of the total variance of returns.

Jumps in volatility are also important as they allows volatility to rapidly increase. For example, in the market stress of Fall 1987, volatility jumped up from roughly 20% to over 50%. Once at this high level, volatility mean reverts back to its long run level, which shows the persistent effect of jumps in volatility on the distribution of returns. We find little, if any, misspecification in the models with jumps in volatility. This provides formal evidence supporting Bates (2000), Duffie, et al. (2000) and Pan (2002) who suggest that jumps in volatility may remove the misspecification documented in models with diffusive stochastic volatility.

It is important to note that the presence of jumps in volatility does not eliminate the need for jumps in returns. With both types of jumps, jumps in returns occur less often, but they still play an important role as they generate the large, though infrequently observed, crash-like movements. For example, in both of the models with jumps in volatility and returns, jumps in returns generate more than half of the crash in 1987 while high volatility explains the rest. An analysis of the three periods of market stress in our sample, 1987, 1997 and 1998, indicates that jumps in volatility and returns play a greater role than diffusive stochastic volatility in generating these episodes. This suggests that jump components

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<sup>3</sup>Pan (2002) finds evidence for large jump risk premia.

should command relatively larger risk premia than diffusive ones, as their contribution to periods of market stress is greater.

Specification diagnostics provide insight into exactly why models without jumps in volatility are misspecified. Heston's (1993) square-root stochastic volatility model requires implausibly large shocks to generate the largest observed movements in returns. For example, the square-root model requires almost an 8 standard deviation shock in returns to generate the crash. This is not due to the square-root specification: Jacquier, et al. (2001) find the same misspecification using a more flexible log-variance model.

Diagnostics indicate that a model with diffusive stochastic volatility and only jumps in returns is also misspecified. Estimates indicate that jump times are clustered, evidence in contrast to the constant arrival intensity assumption. For example, in the model with jumps in returns and diffusive stochastic volatility, we estimate three jump arrivals during the week of the crash in 1987. Although the evidence is slightly different, these results reinforce the conclusions of Bakshi, et al. (1997), Bates (2000) and Pan (2002) who, using additional information in option prices, find strong evidence for misspecification in models with diffusive volatility and jumps in returns. The fact that we arrive at the same conclusion using only returns data is not a coincidence. Due to the absolute continuity of changes in probability measures, returns and options data should contain the same information about the sources of risks, although their impact may be altered due to risk premia.

All of the models we consider generate near closed-form option prices, and we next examine how the different factors impact option prices.<sup>4</sup> Using implied volatility as a metric, we find that the two types of jumps induce important differences in both the term structure and cross-section of implied volatilities. Compared to the stochastic volatility model, adding jumps in returns steepens the slope of the implied volatility curves. Jumps in volatility further steepen implied volatility curves and increase implied volatility for in-

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<sup>4</sup>Liu, Longstaff and Pan (2002) consider related models with jumps in returns and/or jumps in volatility and find that these factors have important implications for optimal portfolio allocation.

the-money options. This latter effect found by Duffie, et al. (2000) and Pan (2002) was labeled as the “hook” or “tipping at the end” effect. Although the motivation for jumps in volatility was to improve on the dynamics of volatility, the results indicate that jumps in volatility also have an important cross-sectional impact on option prices.

Finally, we evaluate the effect of estimation risk on option prices. We decompose estimation risk into two components: parameter and spot volatility estimation risk. These sources of estimation risk have very different effects. For short maturity options, volatility uncertainty is primarily an at-the-money effect and has little impact on out-of-the-money options. Parameter uncertainty, on the other hand, has little at-the-money effect for short maturity options, but generates nearly all of the out-of-the-money uncertainty. For long maturity options, parameter uncertainty dominates as the uncertainty regarding the average level of volatility, speed of mean reversion in volatility and volatility of volatility play a more important role. Spot volatility uncertainty has little impact on long dated options.

The rest of the paper proceeds as follows. Section I introduces the models and Section II describes our estimation approach. Section III summarizes the empirical results for the S&P 500 and Nasdaq 100 indices. Section IV analyzes the option pricing implications and Section V concludes.

## **I Jump-Diffusion Models of Returns and Volatility**

A number of recent papers examine equity price models with jumps in returns and stochastic volatility (see Bakshi, et al. (1997), Bates (2000), Andersen, et al. (2002), Chernov, et al. (2002) and Pan (2002)). While it is clear that both stochastic volatility and jumps in returns are important components, Bakshi, et al. (1997), Bates (2000) and Pan (2002) find strong evidence for misspecification in the volatility process.

Specifically, Bakshi, et al. (1997), using a test developed in Bates (1996), find that the implied structural volatility parameters are inconsistent with time series estimates us-

ing implied volatilities. Additionally, Bates (2000) and Pan (2002) find that the higher moments of volatility changes are inconsistent with the diffusion specification. Together, these results point to the presence of an additional, rapidly moving factor driving conditional volatility which, unlike jumps in returns, has a persistent component.<sup>5</sup> Jumps in volatility provide such a factor.

We assume that the logarithm of asset's price,  $Y_t = \log(S_t)$ , solves

$$\begin{pmatrix} dY_t \\ dV_t \end{pmatrix} = \begin{pmatrix} \mu \\ \kappa(\theta - V_{t-}) \end{pmatrix} dt + \sqrt{V_{t-}} \begin{pmatrix} 1 & 0 \\ \rho\sigma_v & \sqrt{(1-\rho^2)}\sigma_v \end{pmatrix} dW_t + \begin{pmatrix} \xi^y dN_t^y \\ \xi^v dN_t^v \end{pmatrix} \quad (1)$$

where  $V_{t-} = \lim_{s \uparrow t} V_s$ ,  $W_t$  is a standard Brownian motion in  $\mathbb{R}^2$ ,  $N_t^y$  and  $N_t^v$  are Poisson processes with constant intensities  $\lambda_y$  and  $\lambda_v$ , and  $\xi^y$  and  $\xi^v$  are the jump sizes in returns and volatility, respectively.<sup>6</sup> We assume that the parameters and initial conditions have sufficient regularity for the solution of (1) to be well defined.

This specification nests many of the popular models used for option pricing and portfolio allocation applications. Without jumps,  $\lambda_y = \lambda_v = 0$ , (1) reduces to Heston's (1993) square-root stochastic volatility model, the SV model. Bates' (1996) SVJ model has normally distributed jumps in returns,  $\xi^y \sim N(\mu_y, \sigma_y^2)$ , but no jumps in volatility,  $\lambda_v = 0$ . Duffie, et al. (2000) introduced the models with jumps in volatility. The SVIJ model has independently arriving jumps in volatility,  $\xi^v \sim \exp(\mu_v)$ , and jumps in returns,  $\xi^y \sim N(\mu_y, \sigma_y^2)$ . The SVCJ model has contemporaneous arrivals,  $N_t^y = N_t^v = N_t$ , and correlated jump sizes,  $\xi^v \sim \exp(\mu_v)$  and  $\xi^y | \xi^v \sim N(\mu_y + \rho_J \xi^v, \sigma_y^2)$ .

In the SV and SVJ models,  $\theta$  controls the long run mean of  $V_t$  as  $\bar{V} = E(V_t) = \theta$ .

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<sup>5</sup> Andersen, et al. (2001) and Alizadeh, et al. (2002) argue, for U.S. equities and a number of foreign exchange rates, that there are two factors generating volatility, one highly persistent and slowly moving, the other rapidly moving. This behavior is nicely captured in the models we consider with volatility driven by diffusive and jump components.

<sup>6</sup>We originally included a variance risk premia term in the return drift,  $\mu + \beta V_t$ . It was insignificant and was therefore dropped from the analysis, consistent with Andersen, et al (2002) and Pan (2002) who also found this parameter insignificant in models with jumps in returns.

Table I: This table provides summarizes the instantaneous conditional moments for the four models under consideration. In the case of the SVCJ model, the second moment of the jump sizes is  $E[(\xi^y)^2] = \mu_y^2 + 2\mu_y\mu_v\rho_J + \rho_J^2\mu_v^2 + \sigma_y^2$ .

|                                 | SV                 | SVJ                                     | SVCJ  | SVIJ                                    |
|---------------------------------|--------------------|---|---|---|
| $\frac{1}{dt}var_t(dY_t)$       | $V_t$              | $V_t + \lambda_y(\mu_y^2 + \sigma_y^2)$ | $V_t + \lambda_y E((\xi^y)^2)$                | $V_t + \lambda_y(\mu_y^2 + \sigma_y^2)$ |
| $\frac{1}{dt}var_t(dV_t)$       | $\sigma_v^2 V_t$   | $\sigma_v^2 V_t$                        | $\sigma_v^2 V_t + \lambda_y \mu_v$            | $\sigma_v^2 V_t + \lambda_v \mu_v$      |
| $\frac{1}{dt}cov_t(dY_t, dV_t)$ | $\rho\sigma_v V_t$ | $\rho\sigma_v V_t$                      | $\rho\sigma_v V_t + \lambda_y \rho_J \mu_v^2$ | $\rho\sigma_v V_t$                      |

However, in the presence of jumps in volatility,  $\theta$  is only the diffusive, and not total, long run mean of  $V_t$ . With jumps in volatility,  $E(V_t) = \theta + \frac{\mu_v \lambda_v}{\kappa}$  is the long run mean.<sup>7</sup> To see how jumps in returns and volatility affect the second moments of returns and volatility, Table I provides the instantaneous variance and covariance of  $Y_t$  and  $V_t$  for each of the models.<sup>8</sup> The exponentially distributed jumps in  $V_t$  capture the large positive outliers in volatility documented in Bates (2000) and guarantee that  $V_t$  is positive. The SVCJ and SVIJ models have three factors (diffusive stochastic volatility, jumps in returns and jumps in volatility), and we now discuss how these factors impact the distribution of returns.

Although jumps in returns and diffusive stochastic volatility can both generate realistic patterns of *unconditional* non-normalities in returns, they generate very different patterns

<sup>7</sup>To calculate this, integrate the SDE to get

$$E(V_t) = E(V_0) + E\left(\int_0^t \kappa(\theta - V_s) ds\right) + E\left(\int_0^t \sigma_v \sqrt{V_s} dW_s\right) + E\left(\sum_{j=N_0^v}^{N_t^v} \xi_j^v\right).$$

Since the stochastic integral against the Brownian motion is mean zero, Fubini's theorem implies that

$$\bar{V} = \bar{V} + \kappa(\theta - \bar{V})t + E\left[\sum_{j=0}^{\infty} 1_{[N_t^v \leq j]} \xi_j^v\right].$$

Since  $E(\xi_j^v) = \mu_v$  and the jump arrivals are Poisson, we have that  $E\left[\sum_{j=0}^{\infty} 1_{[N_t^v = j]} \xi_j^v\right] = \mu_v \lambda_v t$ , which implies that  $\bar{V} = \theta + \frac{\mu_v \lambda_v}{\kappa}$

<sup>8</sup>The notation  $\frac{1}{dt}var_t(dY_t)$  is a heuristic for  $\lim_{\Delta \rightarrow 0} E\left[(\log(S_{t+\Delta}/S_t))^2 | V_t, S_t\right]$ . Since our initial conditions, drift, diffusion, jump intensity and jump size distributions have sufficient regularity, the limiting operation is valid.

of *conditional* non-normalities. Jumps in returns result in a discrete mixture of normal distributions for returns, which easily generates unconditional and conditional non-normalities over short frequencies such as daily or weekly. Over longer intervals, a central limit effect results in decreasing amounts of excess skewness and kurtosis (see Duffie and Pan (1997) or Das and Sundaram (1999)).

Diffusive stochastic volatility models induce a different distributional structure. Conditional on the current level of volatility, returns are approximately normally distributed over short time horizons such as daily or even weekly. Das and Sundaram (1999) find that diffusive stochastic volatility models, with reasonable parameters, can generate realistic amounts of *conditional* non-normalities in returns only over longer horizons, such as months or even years. This explains why diffusive stochastic volatility models (square-root or otherwise) generate very flat implied volatility curves for short dated options with plausible parameters (see Das and Sundaram (1999) or Jones (2002)).

Models with only jumps in returns and diffusive stochastic volatility can generate realistic patterns of both unconditional and conditional non-normalities, but they have difficulty capturing the dynamics of the conditional volatility of returns. In the SVJ model, the conditional variance of returns is  $V_t + (\mu_y^2 + \sigma_y^2) \lambda_y$  (see Table I). When  $V_t$  is a diffusion, the conditional volatility of returns is time-varying and persistent, but moves slowly as it is driven by normally distributed shocks. As pointed out by Bates (2000) and Pan (2002), this creates misspecification as they find that volatility needs to rapidly increase.

Jumps in volatility provide a factor that combines features from both jumps in returns and diffusive stochastic volatility. Like jumps in returns and unlike stochastic volatility, jumps in volatility are a rapidly moving factor driving returns. Like diffusive stochastic and unlike jumps in returns whose impact on returns is transient, a jump in volatility persists. Thus, jumps in volatility provide a rapidly moving but persistent factor driving volatility. The fact that each factor generates very different behavior is helpful for econometric identification.

There are alternative explanations for the failure of models with only jumps in returns and square-root volatility. These include additional square-root volatility factors, more flexible parametric single-factor stochastic volatility specifications or combinations of these two. More flexible single-factor specifications such as the log and CEV models allow for the volatility of volatility to be state-dependent, a property absent in square-root models. These models are limited because they do not provide closed-form option prices. Andersen, et al. (2002) and Chernov, et al. (2002) find that the log-volatility and square-root models provide a near identical fit to the data and that neither model can capture the fat tails in the return distribution. Chacko and Viceira (2001) and Jones (2002) consider a CEV stochastic volatility specification. While more flexible than the log model, returns in the CEV model are still conditionally normal over short time intervals. Chacko and Viceira (2002) find nonlinearities in the variance of  $V_t$ , but the effect disappears when jumps in returns are added. Consistent with the arguments above, Jones (2002) shows that the CEV model generates realistic unconditional non-normalities, but over short time intervals, the model offers only a modest improvement over the square-root model in generating conditional non-normalities.

Bates (2000) and Chernov, et al. (2002) consider and reject two-factor square-root stochastic volatility models. In fact, Bates (2000, *p.* 218) argues that “the postulated square-root process for implicit factor evolution is fundamentally misspecified,” that the “two-factor models have even greater difficulties than the one-factor models in generating sample paths consistent with the postulated process and the implicit parameters” (*p.* 214) and additionally argues that these models are overfit.

To investigate any improvements generated by additional volatility factors, we estimated a two-factor independent square-root stochastic volatility model where  $dV_{t,i} = \kappa_i (\theta_i - V_{t,i}) dt + \sigma_i \sqrt{V_{t,i}} dW_t^i$ .<sup>9</sup> Consistent with the findings of Bates (2000) and Chernov, et al. (2002), we found little evidence that this model provided any substantive improvement

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<sup>9</sup>We thank the referee for suggesting this.

over the single-factor square-root model. In fact, it suffered from the same misspecification as the square-root and log-volatility models: it cannot capture the tails of the return distribution. This is not surprising as the multi-factor model is instantaneously Gaussian, conditional on total volatility.<sup>10</sup> For these reasons, we focus on a single-factor square-root model of stochastic volatility and extensions incorporating jumps in returns and volatility.

## II Estimating Stochastic Volatility Jump-Diffusions

This section develops a likelihood based estimation approach for estimating multivariate jump-diffusion models using Markov Chain Monte Carlo (MCMC) methods.<sup>11</sup> This approach has four advantages over other estimation methods:<sup>12</sup> (1) MCMC provides estimates of the latent volatility, jump times and jumps sizes; (2) MCMC accounts for estimation risk; (3) MCMC methods have been shown in related settings to have superior sampling properties to competing methods<sup>13</sup> and (4) MCMC methods are computationally efficient so that we can check the accuracy of the method using simulations.

Our approach uses only returns data to estimate and test the models, although it can be extended in a straightforward manner to include option price data (see Eraker (2002)).

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<sup>10</sup>For example, in a model with two independent square root volatility factors, the instantaneous distribution of price changes,

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_{t,1}} dW_{t,1}^s + \sqrt{V_{t,2}} dW_{t,2}^s$$

is still Gaussian:

$$\frac{dS_t}{S_t} \sim N(\mu dt, (V_{t,1} + V_{t,2}) dt)$$

<sup>11</sup>See Robert and Casella (1999) for a general discussion of these methods and Johannes and Polson (2002) for an overview of MCMC estimation of continuous-time models.

<sup>12</sup>Other methods that have been used to estimate models with stochastic volatility and jumps include EMM, simulated maximum likelihood (Brandt and Santa-Clara (2002), Durham and Gallant (2001), Piazessi (2001), calibration (Bates (1996, 2000), Bakshi, et al (1997)) and the implied-state GMM method of Pan (2002).

<sup>13</sup>Jacquier, et al. (1994) find in simulations that MCMC outperforms GMM and QMLE in estimation of stochastic volatility models and Andersen, et al (1999) find that MCMC outperforms EMM.

Due to the absolute continuity of the change in measure from objective to risk-neutral, the presence of jumps in returns or volatility under one measure implies their presence under the other, although due to risk premia, their impact may be distorted. Thus, for specification analysis, returns data should lead to the same conclusion as option price data.

The main advantage of using only returns data is more pragmatic: analyses using option price data tend to use relatively short time spans. For example, Bakshi, et al. (1997) use data from 1988-1991 while Pan (2002) uses data from 1989-1996. This is especially important when estimating models with jumps, which we expect to occur infrequently. Longer samples spanning periods of market stress (the episodes in 1987, 1997 and 1998) will provide more accurate parameter estimates and better insights into the relative roles played by jumps and stochastic volatility.

The basis for our MCMC estimation is a time-discretization of (1)

$$\begin{aligned} Y_{(t+1)\Delta} - Y_{t\Delta} &= \mu\Delta + \sqrt{V_{t\Delta}\Delta}\varepsilon_{(t+1)\Delta}^y + \xi_{(t+1)\Delta}^y J_{(t+1)\Delta}^y \\ V_{(t+1)\Delta} - V_{t\Delta} &= \kappa(\theta - V_{t\Delta})\Delta + \sigma_v\sqrt{V_{t\Delta}\Delta}\varepsilon_{(t+1)\Delta}^v + \xi_{(t+1)\Delta}^v J_{(t+1)\Delta}^v \end{aligned} \quad (2)$$

where  $J_{(t+1)\Delta}^k = 1$  ( $k = y, v$ ) indicates a jump arrival,  $\varepsilon_{(t+1)\Delta}^y$  and  $\varepsilon_{(t+1)\Delta}^v$  are standard normal random variables with correlation  $\rho$  and  $\Delta$  is the time-discretization interval (1-day).<sup>14</sup> The jump sizes retain their distributional structure and the jump times are Bernoulli random variables with constant intensities,  $\lambda_y\Delta$  and  $\lambda_v\Delta$ . This procedure could introduce a discretization bias, although the bias is typically quite small with daily data. We provide simulations below to support this claim.

In this section, we focus on the SVCJ model, as it has the most complicated distributional structure. The posterior distribution summarizes the sample information regarding

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<sup>14</sup>Our framework is not limited to the case in which the discretization interval,  $\Delta$ , equals the observed frequency. To reduce any bias, we could introduce additional unobserved datapoints between dates  $t$  and  $t + 1$ , and treat them as missing datapoints to be included in the MCMC simulation, see Eraker (2001).

the parameters,  $\Theta$ , and the latent volatility, jump times and jump sizes:

$$p(\Theta, J, \xi^y, \xi^v, V|Y) \propto p(Y|\Theta, J, \xi^y, \xi^v, V)p(\Theta, J, \xi^y, \xi^v, V), \quad (3)$$

where  $J$ ,  $\xi^y$ ,  $\xi^v$ ,  $V$  and  $Y$  are vectors containing the time series of the relevant variables. The posterior combines the likelihood,  $p(Y|\Theta, J, \xi^y, \xi^v, V)$ , and the prior,  $p(\Theta, J, \xi^y, \xi^v, V)$ .

An advantage of our approach is the ability to formally incorporate prior information. The need for this is not unique to our approach, but is common in estimating models with jumps. Honore (1998) shows that without prior parameter restrictions a time-discretization of Merton’s (1976) jump-diffusion model generates an unbounded likelihood function. Moreover, the prior contains information about both the parameters and the structure of the latent processes: the stochastic specifications of the jump sizes, jump times and volatility. This reinforces the link between parameters and model specification that is often heuristically used to motivate the presence of jumps. Typically, jumps are described as large, but infrequent movements in returns. This is a form of prior information as the parameters are assumed to induce infrequent but relatively large movements (low  $\lambda_y$  and large  $\mu_y$  and/or  $\sigma_y$ ), as opposed to frequent but small jumps.

Our priors are always consistent with the intuition that jumps are “large” and infrequent. More specifically, we choose a prior on  $\sigma_y$  that places low probability on the jump sizes being small, say less than 1%. For  $\lambda_y$ , our prior places low probability on the daily jump probability being greater than 10% and we place an uninformative prior on  $\mu_y$ . For the other parameters, we specify extremely uninformative priors. For  $\mu$ ,  $\kappa$ ,  $\theta$  and  $\rho_J$ , we use mean zero normal priors with large variances and the prior on  $\rho$  is uniform over  $[-1, 1]$ . Appendix A provides further details on the priors. It is important to note that we impose very little information through our priors.

As the posterior distribution is not known in closed-form, our MCMC algorithm gener-

ates samples by iteratively drawing from the following conditional posteriors:

$$\begin{aligned}
 \text{parameters} &: p(\Theta_i | \Theta_{-i}, J, \xi^y, \xi^v, V, Y), i = 1, \dots, k \\
 \text{jump times} &: p(J_{t\Delta} = 1 | \Theta, \xi^y, \xi^v, V, Y), t = 1, \dots, T \\
 \text{jump sizes} &: p(\xi_{t\Delta}^y | \Theta, J_{t\Delta} = 1, \xi^v, V, Y), t = 1, \dots, T \\
 &: p(\xi_{t\Delta}^v | \Theta, J_{t\Delta} = 1, V, Y), t = 1, \dots, T \\
 \text{volatility} &: p(V_{t\Delta} | V_{(t+1)\Delta}, V_{(t-1)\Delta}, \Theta, J, \xi^y, \xi^v, Y), t = 1, \dots, T
 \end{aligned}$$

where  $\Theta_{-i}$  denotes the elements of the parameter vector except  $\Theta_i$ . Drawing from these distributions is straightforward, with the exception of volatility, as the distribution is not of standard form (Appendix A provides details). The algorithm produces a set of draws  $\{\Theta, J, \xi^y, \xi^v, V\}_{g=1}^G$  which are samples from  $p(\Theta, J, \xi^y, \xi^v, V | Y)$ . Johannes and Polson (2002) provides a review of the theory behind MCMC algorithms.

## A Estimating Volatility and Jumps

For specification analysis and to identify the relative importance of jumps and volatility, we require estimates of the latent volatility, jump times and jump sizes. With continuous record observations all are all observed, but with discretely sampled observations it is not obvious how to separate out the effects of jumps and time-varying volatility. For example, is a large movement in returns generated by a jump in returns or by high volatility? Standard latent variable estimation methods such as the Kalman filter do not apply as our model is neither linear nor Gaussian.

Our MCMC approach provides a straightforward method to estimate the volatilities, jump times and jump sizes by computing the posterior expectation of these variables. We therefore provide a Monte Carlo solution to the classical latent variable estimation problem. The key is that our MCMC algorithm generates samples of the spot volatilities, jump

times and jump sizes, drawn from the joint posterior distribution. Given these samples, the Monte Carlo estimate of the mean of the posterior volatility distribution, for example, is  $E[V_{t\Delta}|Y] \approx \frac{1}{G} \sum_{g=1}^G V_{t\Delta}^g$ , where  $V_{t\Delta}^g$  is the variance at time  $t$  in the  $g^{th}$  iteration of the algorithm. No additional calculations are required: latent variable estimation is just a by-product of our algorithm. Jump time and size estimates are similarly calculated.

These estimates take into account parameter uncertainty. To see this, note that we estimate  $E[V_{t\Delta}|Y]$  and not  $E[V_{t\Delta}|Y, \hat{\Theta}]$ . The former distribution integrates out all of the parameter uncertainty. The latter distribution treats the parameter estimates as known, ignoring the fact that they are random variables.

## B Model Diagnostics and Specification Tests

The spot volatility, jump time and jump size estimates generate a number of informal diagnostics that are useful in assessing the ability of the various models to fit the observed data. For example, consider the return residuals:

$$\frac{Y_{(t+1)\Delta} - Y_{t\Delta} - \mu\Delta - J_{t+1}^y \xi_{t+1}^y}{\sqrt{V_{t\Delta}\Delta}} = \varepsilon_{(t+1)\Delta}^y \approx N(0, 1). \quad (4)$$

It is easy to compute the posterior of these residuals using the parameter and latent variable samples. These residuals need not be exactly normally distributed (since we use a time-discretization of the original model), but they should be approximately normal. Extremely large residuals suggest misspecification as the model requires abnormally large shocks to fit the observed data. Jump time and size estimates provide further diagnostic tools. For instance, any evidence of clustered jump times contradicts the i.i.d. arrival assumption. Jumps on neighboring days of opposite signs, the reversal effect of Schwert (1990), is also evidence against the i.i.d. jump arrivals and sizes.

We also compute formal specification tests. Unlike standard tests that lead to an overall evaluation of the fit of the model (e.g. omnibus Chi-Square tests), we compare the marginal

likelihoods of the models. The advantage of this approach is that it does not rely on large sample distribution theory and provides an intuitive approach to evaluating the relative merits of competing models.

Our approach for comparing nested models is similar to that considered in Jacquier and Polson (1999). Consider testing SV versus SVJ. If we assume positive prior odds on the models,  $p[SV]/p[SVJ] > 0$  (prior ignorance sets this to 1), Bayes theorem implies that the posterior odds are

$$\frac{p(SV|Y)}{p(SVJ|Y)} = \frac{p(Y|SV)}{p(Y|SVJ)} \frac{p(SV)}{p(SVJ)} \quad (5)$$

which is known as the Bayes Factor. Note that this analysis does not assume that the models are necessarily exhaustive, i.e. that  $p[SV] + p[SVJ] = 1$ . Assuming prior ignorance, the Odds Ratio,  $\frac{p(Y|SV)}{p(Y|SVJ)}$ , is interpreted in the following manner: a risk neutral bookmaker would lay odds,  $\frac{p(Y|SV)}{p(Y|SVJ)}$  to 1, on model SVJ versus SV. An advantage of this approach is that the test results in a single number which is the relative odds of the models given the data and there is no appeal to approximate limiting distributions or significance levels. Appendix B derives the marginal likelihoods and Bayes Factors for the models under consideration and provides MCMC estimators.

The fact that the models with jumps in volatility are not formally nested provides no problem for computing the Bayes Factors, e.g., SVCJ *vs.* SVIJ. We compute the Bayes Factor for the non-nested models using the fact that

$$\frac{p(SVCJ|Y)}{p(SVIJ|Y)} = \frac{p(SVCJ|Y)}{p(SVJ|Y)} \frac{p(SVJ|Y)}{p(SVIJ|Y)}. \quad (6)$$

One caveat is that the Monte Carlo standard errors may be larger than in the case of the simple nested comparisons, as the Odds Ratio is now the product of two Odds Ratios.

Following Kass and Raftery (1995), we use the following scale to interpret the Bayes Factors. Evidence against a model is positive if the log Odds Ratio is between 2 and 6; strong if it is been 6 and 10; and very strong if it is greater than 10. It is important to note

that Odds Ratios do not necessarily favor more complex models as they contain a penalty for using more parameters (due to their marginal nature). Because of this, Odds Ratios are often referred to as an “automatic Occam’s razor,” see Smith and Spiegelhalter (1980).

## C Simulation results

We performed Monte Carlo simulations to check the reliability of our estimation approach. This is important for two reasons. First, since we time-discretize the continuous-time model, it is important to check that this does not introduce any biases in parameter estimates. Second, methods for estimating multivariate jump diffusion models are not well developed and it is important to verify that we can reliably estimate the parameters for the given sample size.

Appendix C describes our simulation study and Tables VII and VIII summarize the results. The results indicate that our procedure provides accurate inference. Some parameters are estimated less precisely than others, for example  $\kappa$ , but all are close to their true values. The results also indicate that our priors are not informative, as we use the same priors for parameters common in both the SVJ and SVCJ models, even though the estimation results and true parameters differ.

## III Empirical Results

We estimate the models using S&P 500 and Nasdaq 100 index returns from 1/2/1980-12/31/1999 and 9/24/1985-12/31/1999, respectively. Excluding weekends and holidays, we have 5054 daily observations for the S&P and 3594 observations for the Nasdaq. Table II provides summary statistics for the continuously compounded returns, scaled by 100. In this section, we discuss the estimation results for the S&P 500 for each of the models, how the Nasdaq estimates differ from the S&P and the role of jumps and volatility in the three periods of market stress in our sample.

Table II: This table provides summary statistics for daily return data on the S&P 500 from 1/2/1980-12/31/1999 and the Nasdaq 100 from 9/24/1985-12/31/1999.

|            | Mean    | Volatility | Skewness | Kurtosis | Min      | Max    |
|------------|---------|------------|----------|----------|----------|--------|
| S&P 500    | 13.0185 | 15.8838    | -2.6064  | 62.0328  | -22.8330 | 8.7089 |
| Nasdaq 100 | 24.5841 | 23.2182    | -0.7287  | 11.9424  | -16.3405 | 9.7984 |

## A S&P 500

The second column of Table III provides parameter posterior means and standard deviations for the SV model.<sup>15</sup> The left-hand panel of Figure 1 provide spot volatility estimates over two time periods, 1987-1989 and 1997-1999. The parameter estimates are consistent with previous findings. The average annualized volatility,  $\sqrt{252 \cdot \theta}$ , is 15.10% and is close to the sample volatility of 15.89%. Our estimate of  $\rho$ ,  $-0.40$ , is close to the estimate obtained by Jacquier, et al. (2001) in a log-volatility model ( $\rho = -0.39$ ) and Andersen, et al. (2002) ( $\rho = -0.38$ ), but is slightly smaller than those obtained from studies using option price data (Bakshi, et al. (1997), Bates (2000) and Pan (2002) obtain estimates around  $-0.5$ ).

The SV model is misspecified. This can be seen in a number of ways. First, consider the QQ or normal probability plot of the residuals in the upper left panel in Figure 3. Note the extreme non-normality of the residuals, strong evidence of misspecification. Why does the model require such large shocks? Consider the crash in 1987. On the day of the crash,  $\sqrt{V_t}$  was about 3% which implies that an almost 8 standard deviation return shock is needed to deliver the -23% move. This type of misspecification was also noted in stochastic volatility models by Jacquier, et al. (2001), Andersen, et al. (2002) and Chernov, et al.

<sup>15</sup>For the SV, SVJ and SVCJ models, the MCMC algorithm appears to converge quickly. We discard the first 10,000 iterations as a “burn-in” period and use the last 90,000 to form the Monte Carlo estimates. For the SVIJ model, the algorithm appears to converge more slowly and thus we ran it for 200,000 iterations, discarding the first 10,000 as a “burn-in” period. For the Nasdaq 100, the algorithm converged quickly for all models and thus we discarded the first 10,000 draws and used the last 90,000 draws.

(2002). Second, spot volatility increased for 37 consecutive days prior to the crash in 1987. This is extremely unlikely as the mean reverting drift was exerting downward pressure on volatility throughout this period. Finally, the Bayes Factors reported in Table VI provides evidence against the SV model in favor of the other models.

The third column in Table III provides parameter estimates for the SVJ model. Adding jumps in returns has the expected effect of reducing the demands on the volatility process. For example, average volatility falls from 15.10% to 14.32% and both  $\sigma_v$  and  $\kappa$  fall dramatically, indicating a less volatile, more persistent volatility process. Estimates of spot volatility in Figure 1 show that jumps in returns reduce spot volatility dramatically during periods of market stress, as jumps generate the largest movements. Also note that after the crash in 1987 spot volatility in the SV model remained higher than in the SVJ model for a long time. Thus the effect of omitting jumps in returns lingers after the jump arrived through its impact on estimated volatility.

In the SVJ model, jumps in returns are infrequent events (about 1.5 per year), tend to be negative and are large relative to normal day-to-day movements. A 3 standard deviation jump move is about -15%. Table V decomposes the total variance of returns into stochastic volatility and jump components. The proportion of variance due to jumps is

$$\frac{E \left[ (\xi_t^y)^2 \right] \lambda_y}{\bar{V} + E \left[ (\xi_t^y)^2 \right] \lambda_y}$$

and is 14.65% in the SVJ model.

The normality plots in the SVJ model are improved and the residuals now have slightly thin tails as jumps in returns capture nearly all of the large movements in returns. Figure 2, however, provides evidence that the SVJ model is misspecified. During the week of the crash in 1987, there were three days on which the estimated jump probabilities were extremely high, indicating a cluster of jumps. Why did this occur? During October 1987,

daily volatility was always less than 2%, implying that a 3 standard deviation move in returns due to volatility was only 6%. Due to this, all of the moves larger than 6% were attributed to jumps in returns. Similarly, in October 1997, there were jumps estimated on neighboring days with opposite signs, the reversal effect of Schwert (1990).<sup>16</sup> The clustering of jump arrivals and size reversals are extremely unlikely given the i.i.d. jump time and size specifications and the infrequent nature of jumps (1.5 per year). This suggests that the SVJ model is misspecified. The Bayes factors in Table VI provide additional evidence against the SVJ model and in favor of the models with jumps in volatility.

In conclusion, the SV and SVJ models suffer from similar problems. Periods of market stress are characterized by a short time period with multiple large movements and neither model can generate these movements. In the SV model, these large movements appear as abnormally large shocks, while in the SVJ model, they appear as clustered jumps. In both cases, diffusive volatility cannot increase rapidly enough to generate these episodes and the models are misspecified.

The fourth column of Table III provides parameter estimates for the SVCJ model and the results indicate that jumps in volatility play an important role. When a jump arrives, volatility increases from  $\sqrt{V_{t-}}$  to  $\sqrt{V_{t-} + \xi_v}$ , and the parameter estimates point toward significant increases. For example, when volatility is 15%, an average size jump increases volatility to 24%. The solid line in the right hand panels of Figure 1 shows the estimated volatility path for the SVCJ model and it is very different from those in the SV and SVJ models.

As jump-driven high volatility generates many of the large moves in the volatility process,  $\sigma_v$  is smaller than in the SVJ model. In the SVCJ model, volatility mean reverts faster as  $\kappa$  is almost double its value in the SVJ model. Jumps still arrive at the rate of about 1.5 per year, but jumps in returns are smaller. Jumps in returns play a lesser role

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<sup>16</sup>Johannes, et al. (1999) document this pattern and develop models with state dependent arrivals and jump sizes to capture this phenomenon.

Table III: Parameter estimates for the S&P 500 index data, 1/2/1980-12/31/1999. The models and parameterizations are given in Section 1 and the estimates correspond to percentage changes in the index value. For each parameter we report the mean of the posterior deviation and the standard deviation of the posterior in parentheses.

|             | SV               | SVJ              | SVCJ             | SVIJ             |
|-------------|------------------|------------------|------------------|------------------|
| $\mu$       | 0.0444 (0.0110)  | 0.0496 (0.0109)  | 0.0554 (0.0112)  | 0.0506 (0.0111)  |
| $\theta$    | 0.9052 (0.1077)  | 0.8136 (0.1244)  | 0.5376 (0.0539)  | 0.5585 (0.0811)  |
| $\kappa$    | 0.0231 (0.0068)  | 0.0128 (0.0039)  | 0.0260 (0.0041)  | 0.0250 (0.0057)  |
| $\sigma_v$  | 0.1434 (0.0128)  | 0.0954 (0.0104)  | 0.0790 (0.0074)  | 0.0896 (0.0115)  |
| $\mu_y$     |                  | -2.5862 (1.3034) | -1.7533 (1.5566) | -3.0851 (3.2485) |
| $\rho_J$    |                  |                  | -0.6008 (0.9918) |                  |
| $\sigma_y$  |                  | 4.0720 (1.7210)  | 2.8864 (0.5679)  | 2.9890 (0.7486)  |
| $\mu_v$     |                  |                  | 1.4832 (0.3404)  | 1.7980 (0.5737)  |
| $\rho$      | -0.3974 (0.0516) | -0.4668 (0.0579) | -0.4838 (0.0623) | -0.5040 (0.0661) |
| $\lambda_y$ |                  | 0.0060 (0.0021)  | 0.0066 (0.0020)  | 0.0046 (0.0020)  |
| $\lambda_v$ |                  |                  |                  | 0.0055 (0.0032)  |

in the SVCJ model than in the SVJ model as the proportion of total variance coming from jumps in returns is now only 9.96%. The jump size estimates in the upper right hand panel of Figure 2 indicate that nearly all of the jumps are negative, as the increased volatility after a jump generates the reversal effect. The estimate of  $\rho$  falls again, which is consistent with Duffie, et al. (2000).

The normal residual plot in Figure 3 gives no indication of misspecification in the SVCJ model. As jumps in returns are smaller, they explain less of the large movements and the Brownian increments are almost perfectly normally distributed. Although it is difficult to see in Figure 2, there was only a single jump during the week of the crash in 1987 on

October 19<sup>th</sup>, when returns had a negative jump of -14% and volatility jumped upward from 40% to just over 50%. The Bayes factors in Table VI support SVCJ over the SV and SVJ models.

Figure 1 shows that the SVIJ and SVCJ models exhibit very similar behavior as spot volatility in the two models is almost identical. The jump times and sizes are also similar, and are therefore not reported. Allowing volatility to jump independently of returns provides additional flexibility over the SVCJ model, although the model is harder to estimate as jumps in volatility are not signalled by a jump in returns. This added flexibility allows volatility to play an even more important role than in the SVCJ model as the proportion of variance due to jumps in returns is less than in the SVJ or SVCJ models. Diagnostics do not provide any evidence of misspecification in the SVIJ model. The Bayes Factor comparing SVCJ and SVIJ is over 10, evidence that favors the SVIJ model. A caveat is in order: the SVIJ model is more difficult to estimate as the jumps do not occur simultaneously and this results in greater posterior parameter uncertainty for all of the jump size parameters.

## B Nasdaq 100

Table IV provides the Nasdaq parameters estimates, using the same priors that were used for the S&P and Figures 4 and 5 provide volatility and jump estimates. The biggest difference between the Nasdaq and S&P returns is that Nasdaq’s volatility is much higher and more volatile than S&P volatility. For example, in the SVJ model, the average daily variance for the Nasdaq is 1.91, more than double the average for the S&P, 0.91. For each of the models,  $\sigma_v$  is roughly 50% higher for the Nasdaq than for the S&P and the speed of volatility mean reversion is also higher for the Nasdaq.

Although volatility is systematically higher, this does not imply that jumps play a lesser role, in fact, the contrary is true. Jumps arrive about three times more often, although the sizes are typically smaller. Since the largest single day move in the Nasdaq is smaller than in the S&P (-16% compared to -23%), jumps in returns need not generate as many extremely

Table IV: Parameter estimates for the four models under consideration using Nasdaq 100 index data, 9/24/1985-12/31/1999. The models and parameterizations are given in Section 1 and the estimates correspond to percentage changes in the index value. For each parameter we report the mean of the posterior deviation and the standard deviation of the posterior in parentheses

|             | SV               | SVJ              | SVCJ             | SVIJ             |
|-------------|------------------|------------------|------------------|------------------|
| $\mu$       | 0.1070 (0.0196)  | 0.1240 (0.0205)  | 0.1284 (0.0206)  | 0.1164 (0.0196)  |
| $\theta$    | 2.0242 (0.2309)  | 1.9067 (0.2853)  | 0.9249 (0.1547)  | 1.0593 (0.1506)  |
| $\kappa$    | 0.0264 (0.0072)  | 0.0176 (0.0052)  | 0.0414 (0.0106)  | 0.0371 (0.0080)  |
| $\sigma_v$  | 0.2155 (0.0226)  | 0.1692 (0.0190)  | 0.1216 (0.0188)  | 0.1395 (0.0196)  |
| $\mu_y$     |                  | -2.4755 (1.0276) | -1.8868 (0.7407) | -2.6231 (2.5946) |
| $\rho_J$    |                  |                  | -0.0993 (0.1692) |                  |
| $\sigma_y$  |                  | 2.0788 (0.4375)  | 1.8452 (0.3079)  | 2.0389 (0.4485)  |
| $\mu_v$     |                  |                  | 2.1054 (0.3989)  | 2.5227 (0.5946)  |
| $\rho$      | -0.2869 (0.0553) | -0.3366 (0.0695) | -0.3427 (0.0975) | -0.3910 (0.0801) |
| $\lambda_y$ |                  | 0.0172 (0.0097)  | 0.0202 (0.0074)  | 0.0081 (0.0044)  |
| $\lambda_v$ |                  |                  |                  | 0.0140 (0.0058)  |

large moves, see Figure 5. The combination of the slightly lower jump size volatility with the higher arrival rate results in the proportion of total variance due to jumps increasing slightly, except for the SVIJ model, see Table V. An easier way to see how the volatility structure of the S&P and Nasdaq indices differ is to compare spot volatility estimates in Figure 6. The volatility of the two indices was similar prior to 1990, but since then the volatility of the Nasdaq has been higher, often three times higher than the volatility of the S&P.

The leverage effect,  $\rho$ , is less pronounced for the Nasdaq. This is consistent with the sample statistics, as Nasdaq's skewness is smaller than the S&P. As in the case of the S&P,

Table V: The 2<sup>nd</sup> and 3<sup>rd</sup> columns give the average annualized spot stochastic volatility for each of the models and for the S&P 500 and Nasdaq 100. The 4<sup>th</sup> and 5<sup>th</sup> columns give the average annualized total volatility, which is the sum of stochastic volatility and jumps in returns components. The last two columns gives the proportion of the variance due to jumps.

|      | Spot Volatility |       | Total Volatility |       | Return Jump Variance (% of Total) |       |
|------|-----------------|-------|------------------|-------|-----------------------------------|-------|
|      | S&P             | NDX   | S&P              | NDX   | S&P                               | NDX   |
| SV   | 15.10           | 22.59 | 15.10            | 22.59 | –                                 | –     |
| SVJ  | 14.32           | 21.92 | 15.49            | 24.06 | 14.65                             | 17.00 |
| SVCJ | 15.18           | 22.18 | 15.99            | 23.59 | 9.96                              | 11.63 |
| SVIJ | 15.51           | 22.51 | 16.18            | 23.22 | 8.17                              | 5.99  |

Table VI: Log-Bayes factors for the S&P 500 and Nasdaq 100

|            | SV vs SVJ | SV vs SVCJ | SV vs SVIJ | SVJ vs SVIJ | SVCJ vs SVIJ |
|------------|-----------|------------|------------|-------------|--------------|
| S&P 500    | 47.62     | 49.32      | 59.45      | 33.83       | 10.12        |
| Nasdaq 100 | 27.16     | 28.95      | 56.03      | 33.19       | 27.08        |

$\rho$  increases, in absolute value, from SV to SVIJ. The average size of a jump in volatility in the Nasdaq is larger than in the S&P. However, as Nasdaq volatility is higher than the volatility of the S&P, the proportional increase in volatility due to an average-sized jump is smaller for the Nasdaq. The specification diagnostics, both formal and informal, indicate that models with jumps in volatility are preferred over those without jumps in volatility.

## C Periods of Markets Stress: Jumps and Volatility

The estimates of spot volatility and jumps provide a means to evaluate the contribution of these factors to the three periods of market stress in our sample: October 1987, October

1997 and late Summer-Fall 1998. A priori it is unclear if jumps in returns or excess volatility were responsible for these periods, as both can generate large movements. Understanding how these factors contribute to these periods is important because it is precisely the extreme movements during periods of market stress that may cause investors to demand large premia to carry these risks. We focus on the behavior of volatility and jumps in the SVJ and SVCJ models for the S&P 500 index.

In October 1987, the SVJ model attributes most of the large movements to jumps in returns. Of the 23% decline on the day of the crash, a jump in returns generated 18.8% of the move and there were also two other jumps during the week. However, when allowed to jump, volatility plays a more prominent role in generating market stress. In the SVCJ model, there was a jump about two weeks before the crash which increased volatility from 21% to 36%, and on October 16<sup>th</sup> volatility jumped from 35%-42%. Another jump arrived on the day of the crash, increasing volatility to 50% and delivering a -14% jump in returns. This implies that in the SVCJ model jumps in returns generated more than half of the crash. The large movements after the crash (+8%, -4% and -8%) are attributed to Brownian shocks combined with the high volatility, in contrast to the SVJ model. The SVIJ model delivers similar results, although the exact timing of jumps in volatility and returns are slightly different.

In October 1997, there was a -7% move on the 27<sup>th</sup> and a +5% move on the 28<sup>th</sup>. The SVJ model attributes both moves to jumps in returns and volatility stays constant at about 19%. SVCJ estimates indicate that there was a single jump on the 27<sup>th</sup> which lowered returns by -6% and increased volatility from 16% to 25%. Thus the first move was due to a jump in returns, while the second was due to high volatility, a more plausible explanation than back-to-back jumps in returns. These periods indicate that both jumps in volatility and returns are important components of periods of market stress.

Late Summer and early Fall 1998 was a period with a number of relatively large moves in the 3%-5% range. Our estimates indicate that most of the moves were generated by

high volatility (as opposed to jumps in returns), which, in turn, was generated by jumps in volatility. The first, on July 21, 1998 occurred coincidental to Alan Greenspan's comments on the economy and one day after the *Wall Street Journal* first reported the large losses at Long Term Capital Management. Although the move in the S&P 500 was only -1.6%, in the SVCJ model our algorithm estimated that a jump in volatility occurred with relatively high probability (about 50%) increasing volatility from 12% to just over 20%. Jumps in returns and volatility also occurred at the end of August, corresponding to the halt in trading of the Russian Ruble. Over time, volatility gradually mean-reverted back to relatively normal levels.

These three periods indicate that jumps in volatility and returns are the primary components that generated the periods of market stress, while diffusive stochastic volatility plays a secondary role. This suggests that it may be more appropriate to assign risk premia to jumps in returns and volatility components, rather than diffusive stochastic volatility.

## IV Option Pricing Implications

This section analyzes two option pricing implications: how the models induce different option prices and the effect of parameter and spot volatility estimation risk on option prices. Eraker (2002) provides a more detailed analysis of the option pricing implications.

### A Differences across models

The time series results indicate that jumps in volatility and returns play an important role in determining the dynamics of returns. To evaluate their economic significance, we examine the cross-section and term structure of implied volatilities for the different models, conditional on estimated parameters and spot volatility. These results are comparable to those in Duffie, et al. (2000) with one important difference. Duffie, et al. (2000) calibrate the parameters and spot volatility to minimize option pricing errors on a given day, while

ours use only the information contained in historical returns.

Figure 7 displays implied volatility (IV) curves for the SV, SVJ and SVCJ models for three maturities. The SVCJ and SVIJ models deliver similar curves, so we omitted the SVIJ curves. The IV curves were computed using call option prices conditional on the posterior mean of spot volatility and parameters for various strikes. To frame the results, we also plot market implied volatilities for options traded on October 31, 1994, a randomly selected average volatility day in our sample.<sup>17</sup>

Results indicate that jumps in returns and volatility induce important differences in the shape of the IV curves, especially for short maturities. First, and most apparent, the addition of jumps in returns and jumps in volatility significantly increases the curvature of the IV curves. For short maturity options, the difference between the SV, SVJ and SVCJ IV curves for far in-the-money (ITM) or out-of-the money (OTM) options is quite large. This should not be a surprise as jumps in returns and volatility increase the conditional non-normalities of the underlying distribution. Our results are different from those in Andersen, et al. (2002) and Pan (2002) who require significant jump risk premium to generate differences between the IV curves generated by the SV and SVJ model.

Note the substantial increase in IV for ITM options for the SVCJ model. This hook or “tipping at the end” effect (Duffie, et al. (2000) and Pan (2002)) is not present in the SVJ model. It appears to be an important feature of market implied volatilities, although the SVCJ model, without volatility jump risk premia, does not generate a sharp enough hook to match observed IV curves. Third, the SV model generates very flat IV curves as it does not generate any substantive conditional departures from normality. Last, we find a significant flattening out effect as time-to-maturity increases for all of the models. This occurs because as maturity increases, the fat-tails and asymmetries in the conditional distribution are driven to a larger extent by diffusive volatility, rather than jumps.

To see the impact of adding a mean jump size risk premium, the right hand panel

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<sup>17</sup>We thank the referee for suggesting this.

of Figure 7 displays the IV curves adding a modest 2% mean jump size in returns risk premium. The premium increases the OTM slope of the IV curves, but the effect is small. Casual observation indicates that the risk premium improves the models' ability to fit option prices, especially for longer maturities. The fact that this small risk premium brings model implied prices close to market prices is in contrast to Pan (2002) who estimates the risk premium to be 18%.

## B Estimation Risk and Option Prices

Our MCMC approach, through the posterior distribution, quantifies estimation risk: the uncertainty inherent in estimating parameters and spot volatility. This section examines how this uncertainty impacts option prices. If  $\Theta$  and  $V_t$  were known, the price of a call option struck at  $K$ , with  $\tau$  days to maturity, conditional on the current spot index level,  $S_t$ , is the usual option pricing function,  $C(V_t, \Theta, S_t, K, \tau) = E_t^{\mathbb{Q}} [e^{-r(\tau)} \max(S_{t+\tau} - K, 0)]$ , where  $\mathbb{Q}$  is the pricing measure.

This ignores that  $\Theta$  and  $V_t$  are unknown and must be estimated. In our setting,  $p(\Theta, V_t|Y)$ , which our MCMC algorithm computes, quantifies the uncertainty in estimating  $V_t$  and  $\Theta$ . From the econometrician's perspective, with uncertain spot volatility and parameters, the price of a call option is given by

$$C_t(S_t, K, \tau) = \int C(V_t, \Theta, S_t, K, \tau) p(\Theta, V_t|Y) d\Theta dV_t$$

which integrates out all posterior uncertainty. This assumes that the agents pricing the options know the true volatility and parameters. We do not address the difficult issue of evaluating the impact of parameter uncertainty on equilibrium prices (see, e.g., Anderson, Hansen and Sargent (2000) for one approach to this issue).

To quantify the impact of estimation risk, we compute the posterior distribution of option prices when both  $\Theta$  and  $V_t$  are uncertain (Case 1), when only  $V_t$  is unknown (Case

2) and when only  $\Theta$  is uncertain (Case 3). We do this by integrating out the uncertainty in  $\Theta$  and/or  $V_t$  as summarized by the posterior distribution. In the case where  $\Theta$  or  $V_t$  are uncertain, we condition on the posterior mean for the other quantity. Figure 8 displays the posterior mean and a (10%, 90%) posterior coverage interval for each case, for three maturities for the SVCJ model, which is representative of the other jump models.

A comparison of the graphs in the upper and lower left panels indicates that nearly all of the ATM uncertainty is generated by uncertainty in  $V_t$ . The opposite effect occurs for OTM options. For this case, the middle left panel indicates that parameter uncertainty dominates. Why does this occur? Note that while the posterior means of  $\mu_y$  and  $\sigma_y$  are  $-1.7$  and  $2.9$ , a one-standard deviation symmetric coverage interval is  $(0.2\%, -3.3\%)$  for  $\mu_y$  and  $(2.3\%, 3.5\%)$  for  $\sigma_y$ . Since the tails in the conditional distribution over short intervals load heavily on these parameters, any uncertainty over these parameters has a large impact on OTM options. A similar argument holds for  $\lambda_y$  and  $\mu_v$ .

At medium time horizons,  $V_t$  and  $\Theta$  uncertainty is still slightly greater for ATM and OTM options, but the effect is smaller. At long horizons, a different effect occurs: parameter uncertainty dominates and spot volatility uncertainty has a minimal effect. The intuition for this is clear. The conditional distribution of the index value over a year is largely determined by the long run behavior of the model, which in turn is determined by the parameters driving volatility:  $\theta$ ,  $\kappa$ , and  $\sigma_v$ .

## V Conclusions

This paper analyzed models with jumps in returns and in volatility. For both the S&P 500 and Nasdaq 100 index, results indicate that both of these jump components are important, and models without jumps in volatility are misspecified. Models with only diffusive stochastic volatility and jumps in returns are misspecified because they do not have a component driving the conditional volatility of returns which is rapidly moving.

The information in the time series of returns indicates that jumps in returns and jumps in volatility have a strong impact on option prices. Compared to a model with only jumps in returns, models with jumps in volatility result in a significant increase in IV for deep in-the-money or out-of-the-money options. The fact that jumps in volatility have such a large impact on IV is somewhat surprising as the original motivation for including jumps in volatility was to improve on the dynamics of spot volatility and not necessarily to generate more realistic IV curves, although, of course, these two goals are clearly related.

Parameter and volatility estimation risk also has an important impact on option prices. While parameter uncertainty might result in some option price uncertainty for short maturity at-the-money options, the impact can be as large as  $\pm 2$  implied volatilities for out-of-the-money options. Volatility uncertainty for at-the-money short maturity options leads to prices that differ by  $\pm 2$  implied volatilities, far greater than the bid-ask spread. Although the results on parameter uncertainty in option pricing are new, the importance of volatility uncertainty is not new. Merton (1980) recognized this as an important feature and state that the “most important direction is to develop accurate variance estimation models which take into account of the errors in variance estimates” (*p.* 355). Our MCMC algorithm provides such a method.

Our results indicate that parameter and spot volatility estimation risk is substantial. To obtain more accurate estimates, option price data may be extremely useful. Chernov and Ghysels (2000), Eraker (2002) and Pan (2002) all use spot and option price data to estimate various models. However, while the addition of option prices will aid in the estimation of the spot volatility, it is unclear if it will significantly reduce the parameter uncertainty. This occurs because the risk premia embedded in option prices introduce additional parameters, which are typically difficult to estimate. Unless the risk premia are restricted as in Pan (2002), it is not clear if the aggregate parameter uncertainty will increase or decrease with the addition of option price data.

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## A Posterior distributions for Jumps and Volatility

The conditional posteriors for the jump sizes and jump times are new and are derived as follows. We use the SVCJ model because it is the most complex, in terms of distributional structure. Recall the prior structure for the jump sizes in SVCJ,  $\xi_t^v \sim \exp(\mu_V)$  and  $\xi_t^y | \xi_t^v \sim N(\mu_y + \rho_J \xi_t^v, \sigma_y^2)$ . This specification allows us to use the Gibbs sampler to exploit the conditional independence of the jumps in volatility as it is easy to draw from the conditional posterior. The conditional posterior for the jump sizes to volatility is  $p(\xi_{(t+1)\Delta}^v | J_{(t+1)\Delta} = 1, \Theta, \xi^y, V, Y)$  which, by Bayes rule is proportional to

$$p\left(Y_{(t+1)\Delta}, V_{(t+1)\Delta} | V_{t\Delta}, J_{(t+1)\Delta} = 1, \Theta, \xi_{(t+1)\Delta}^y, \xi_{(t+1)\Delta}^v\right) p\left(\xi_{(t+1)\Delta}^v | J_{(t+1)\Delta} = 1, \Theta, \xi_{(t+1)\Delta}^y\right)$$

where the first term,  $p\left(Y_{(t+1)\Delta}, V_{(t+1)\Delta} | V_{t\Delta}, J_{(t+1)\Delta} = 1, \Theta, \xi_{(t+1)\Delta}^y, \xi_{(t+1)\Delta}^v\right)$ , is a bivariate normal distribution and the second term is, by Bayes rule, proportional to the product of  $p\left(\xi_{(t+1)\Delta}^y | \xi_{(t+1)\Delta}^v, J_{(t+1)\Delta} = 1, \Theta\right)$  and  $p\left(\xi_{(t+1)\Delta}^v | J_{(t+1)\Delta} = 1, \Theta\right)$ . This is a product of a normal and an exponential. To find the full conditional posterior, completing the square for all three terms as a function of  $\xi_{(t+1)\Delta}^v$  leads to a truncated normal

$$p\left(\xi_{(t+1)\Delta}^v | J_{(t+1)\Delta} = 1, \Theta, \xi^y, V, Y\right) \propto 1_{[\xi_{(t+1)\Delta}^v > 0]} N\left(\alpha_{(t+1)\Delta}^v, \omega_{(t+1)\Delta}^v\right)$$

where  $\alpha_{(t+1)\Delta}^v$  and  $\omega_{(t+1)\Delta}^v$  are straightforward to compute. The conditional posterior given a jump is therefore a truncated normal distribution. When  $J_{(t+1)\Delta} = 0$ , the conditional posterior is  $\xi_t^v \sim \exp(\mu_v)$ , as the data provides no information about the jump size.

The posterior for the jumps in returns is similarly derived. Bayes Rule implies that

$$p\left(\xi_{(t+1)\Delta}^y | \xi_{(t+1)\Delta}^v, J_{(t+1)\Delta} = 1, \Theta, V_{t\Delta}, Y_{(t+1)\Delta}\right) \propto p\left(Y_{(t+1)\Delta} | \xi_{(t+1)\Delta}^y, \xi_{(t+1)\Delta}^v, J_{(t+1)\Delta} = 1, \Theta, V_{t\Delta}\right) p\left(\xi_{(t+1)\Delta}^y | \xi_{(t+1)\Delta}^v, \Theta\right).$$

Since both of the densities are Gaussian, we have that

$$\xi_{(t+1)\Delta}^y | \xi_{(t+1)\Delta}^v, J_{(t+1)\Delta} = 1, \Theta, V_{t\Delta}, Y_{(t+1)\Delta} \propto N\left(\alpha_{(t+1)\Delta}^y, \omega_{(t+1)\Delta}^y\right) \quad (7)$$

where  $\alpha_{(t+1)\Delta}^y$  and  $\omega_{(t+1)\Delta}^y$  are easy to compute.

For the jump times, which are assumed to arrive contemporaneously, the posterior combines information from the returns and from the volatility. As  $J_{(t+1)\Delta}$  can take only two values, its posterior is Bernoulli,  $Ber(\lambda_{(t+1)\Delta})$ . To compute the Bernoulli probability, we use the conditional independence of increments to volatility and returns to get that

$$\begin{aligned} p(J_{(t+1)\Delta} = 1 | V_{(t+1)\Delta}, V_{t\Delta}, Y_{(t+1)\Delta}, \xi_{(t+1)\Delta}^y, \xi_{(t+1)\Delta}^v, \Theta) \\ \propto \lambda \Delta \cdot p(Y_{(t+1)\Delta}, V_{(t+1)\Delta} | V_{t\Delta}, J_{(t+1)\Delta} = 1, \xi_{(t+1)\Delta}^y, \xi_{(t+1)\Delta}^v, \Theta) \end{aligned}$$

which, again, is easy to calculate since  $p(Y_{(t+1)\Delta}, V_{(t+1)\Delta} | V_{t\Delta}, J_{(t+1)\Delta} = 1, \xi_{(t+1)\Delta}^y, \xi_{(t+1)\Delta}^v, \Theta)$  is a bi-variate Gaussian density. Computing the conditional posterior for  $J_{(t+1)\Delta} = 0$  proceeds similarly which gives the Bernoulli probability.

The conditional posterior for volatility,  $p(V_{t\Delta} | Y_{t\Delta}, V_{(t+1)\Delta}, V_{(t-1)\Delta}, \xi_{t\Delta}^y, \xi_{t\Delta}^v, J_{t\Delta}, \Theta)$ , is not a known distribution. To sample from it, we use a random-walk Metropolis algorithm (see Johannes and Polson (2002)). The conditional posteriors for the parameters are standard. Given our conjugate priors, the conditional posteriors for  $\{\alpha, \kappa, \theta, \sigma_v, \lambda_y, \lambda_v, \mu_y, \sigma_y^2, \mu_v\}$  are all standard distributions and we omit the derivations as they can be found in standard texts. For  $\rho$  we use an independence Metropolis algorithm with a proposal density centered at the sample correlation between the Brownian increments.

We now discuss our choices of prior distributions and parameters. Wherever possible we choose standard conjugate priors which allows us to directly from the conditional posteriors. Our prior distributions for the parameters are:  $\alpha \sim N(1, 25)$ ,  $\kappa\theta \sim N(0, 1)$ ,  $\kappa \sim N(0, 1)$ ,  $\sigma_v^2 \sim IG(2.5, 0.1)$ ,  $\rho \sim \mathcal{U}(-1, 1)$ ,  $\lambda_y = \lambda_v \sim Beta(2, 40)$ ,  $\mu_y \sim N(0, 100)$ ,  $\sigma_y^2 \sim IG(5.0, 20)$ ,

$\mu_v \sim G(20, 10)$  and  $\rho_J \sim N(0, 4)$  where  $G$  refers to a Gamma distribution,  $IG$  refers to the Inverse Gamma distribution and  $\mathcal{U}$  a standard uniform distribution. All of the prior distributions are uninformative with the exception of  $\sigma_y$  and  $\lambda_y$ . The simulation results given below demonstrate that the information imposed by these priors is minor relative to the information in the likelihood function. This is further demonstrated by the fact that we use the same priors for the Nasdaq and S&P data, which deliver drastically different parameter estimates.

## B Bayes factor calculations

Suppose you wish to compare SV and SVJ. Let  $\Omega = \left\{ \Theta, \{V_{t\Delta}, J_{t\Delta}, \xi_{t\Delta}\}_{t=1}^T \right\}$  be a matrix of latent variables and parameters which gives the marginal likelihoods as

$$p(Y|SV) = \int p(Y|\Omega, SV)p(\Omega|SV)d\Omega \text{ and } p(Y|SVJ) = \int p(Y|\Omega, SVJ)p(\Omega|SVJ)d\Omega.$$

Now as the SVJ model embeds the simple SV model when the entire vector of jump times is zero ( $J = 0$ ), we have the identity that  $p(Y|\Omega, SV) = p(Y|\Omega, J = 0, SVJ)$ . Moreover, if we assume in the priors that  $p(\Omega|SV) = p(\Omega|J = 0, SVJ)$  (parameters in common have the same priors), then

$$p(Y|SV) = \int p(Y|\Omega, J = 0, SVJ)p(\Omega|J = 0, SVJ)d\Omega = p(Y|J = 0, SVJ).$$

Bayes rule also implies that

$$P(J = 0|Y, SVJ) = \frac{P(Y|J = 0, SVJ)}{P(Y|SVJ)}P(J = 0|SVJ) \tag{8}$$

and we have the important identity which we use to develop an MCMC estimator,

$$\text{odds}(sv : svj) = \frac{P(J = 0|Y, SVJ)}{P(J = 0|SVJ)}. \quad (9)$$

This is simply a ratio of prior ordinate to posterior ordinate and the key observation is that this can be directly computed from the MCMC output whereas the marginal likelihoods above are not available (see Jacquier and Polson (1999)).

In the SVJ model, we can compute

$$\begin{aligned} p(J = 0|SVJ) &= \int_0^1 p(J = 0|\lambda, SVJ)p(\lambda|SVJ)d\lambda \\ &= \int_0^1 (1 - \lambda)^T \frac{\lambda^{\alpha_0-1}(1 - \lambda)^{\beta_0-1}}{\mathcal{B}(\alpha_0, \beta_0)} d\lambda = \frac{\mathcal{B}(\alpha_0, T + \beta_0)}{\mathcal{B}(\alpha_0, \beta_0)}. \end{aligned}$$

For the other portion, a similar argument gives

$$p(J = 0|Y, SVJ) = \int_0^1 p(J = 0|\lambda, Y, SVJ)p(\lambda|Y, SVJ)d\lambda \quad (10)$$

which a straightforward computation yields

$$p(J = 0|Y, SVJ) = \int_J \left[ \frac{B(\alpha_0 + \sum_{t=0}^T J_t, \beta_0 + 2T - \sum_{t=0}^T J_t)}{B(\alpha_0 + \sum_{t=1}^T J_t, \beta_0 + T - \sum_{t=1}^T J_t)} \right] p(J|Y)dJ$$

since

$$\begin{aligned} p(\lambda|Y, SVJ) &= \int_J p(\lambda|J, Y, SVJ)p(J|Y)dJ = \int_J p(\lambda|J, SVJ)p(J|Y)dJ \\ &= \int_J \frac{\lambda^{\alpha_0 + \sum_{t=0}^T J_t - 1}(1 - \lambda)^{\beta_0 + T - \sum_{t=1}^T J_t - 1}}{B(\alpha_0 + \sum_{t=1}^T J_t, \beta_0 + T - \sum_{t=1}^T J_t)} p(J|Y)dJ. \end{aligned}$$

Computing this integral with the Monte Carlo samples gives:

$$p(J = 0|Y, SVJ) = \frac{1}{G} \sum_{g=1}^G \frac{\mathcal{B}(\alpha_0 + \sum_{t=0}^T J_t^g, \beta_0 + 2T - \sum_{t=0}^T J_t^g)}{\mathcal{B}(\alpha_0 + \sum_{t=1}^T J_t^g, \beta_0 + T - \sum_{t=1}^T J_t^g)} \quad (11)$$

and we therefore have an MCMC estimator:

$$odds(sv : svj) = \frac{\mathcal{B}(\alpha_0, \beta_0)}{\mathcal{B}(\alpha_0, T + \beta_0)} \frac{1}{G} \sum_{g=1}^G \frac{\mathcal{B}(\alpha_0 + \sum_{t=0}^T J_t^g, \beta_0 + 2T - \sum_{t=0}^T J_t^g)}{\mathcal{B}(\alpha_0 + \sum_{t=1}^T J_t^g, \beta_0 + T - \sum_{t=1}^T J_t^g)}. \quad (12)$$

The computation of the other Odds Ratios is similar. In following the literature, we compute log odds ratios.

## C Simulation experiments

Our simulations used 100 artificial data sets consisting of 4,000 data points. The data were generated using the Euler discretization of the continuous-time model with  $\Delta = 1/20$ . Hence, our estimation method is measured against the artificial data generated by the true continuous time process. We used a posterior sample size of 50,000 for each of the MCMC runs.

The number of data points and the true parameters are suggestive of what one can expect from daily equity price data, although we tried to stack the deck against our methodology by making the jumps extremely unlikely. An intensity of 0.006 indicates that there are only 1.5 jumps per years. With 4000 daily observations, this implies that we essentially only have jumps 26 jumps per sample.

Tables VII and VIII report simulations for the SVJ and SVCJ models. The tables report the means and the root mean squared error (RMSE) and the results indicate that the algorithm delivers extremely accurate estimates for most of the parameters, with the exception of  $\kappa$ , which appears to be slightly biased. This is not necessarily surprising: this

parameter governs the speed of mean reversion of stochastic volatility and is difficult to estimate. Our results are similar to those found reported in Jacquier, et al. (1994) for the stochastic volatility parameters.

In order to see how much of the error in estimating jump components was due to the extremely rare jumps in returns, Table VIII reports the results when we increased the arrival intensity to 0.015 (3.6 jumps per year). As expected, increasing the arrival rate of jumps improves the ability of the algorithm to estimate the parameters of the jump distribution, although the increased noise makes estimation of the stochastic volatility components slightly less accurate. The results for the SVCJ model are similar, although we are not able to reliably estimate the  $\rho_J$  parameter.  $\rho_J$  is essentially a regression parameter between the jumps in returns and volatility,  $\xi^y = \mu_y + \rho_J \xi^v + \varepsilon$  where  $\varepsilon \sim \mathcal{N}(0, \sigma_y^2)$ . Since jumps are latent and jumps are rare events, this is not surprising.

Table VII: This table provides a summary of the simulation results for the SVJ model. 100 sample paths were simulated off the true parameters for and we report the average across the simulations as well as the root mean squared error (RMSE). We consider high and low jump intensity cases.

|                     | $\alpha$ | $\theta$ | $\kappa$ | $\sigma_v$ | $\mu_y$ | $\sigma_y$ | $\rho$  | $\lambda$ |
|---------------------|----------|----------|----------|------------|---------|------------|---------|-----------|
| Low Jump Intensity  |          |          |          |            |         |            |         |           |
| True                | 0.0500   | 0.8000   | 0.015    | 0.1000     | -3.0000 | 3.5000     | -0.4000 | 0.0060    |
| Mean                | 0.0528   | 0.8212   | 0.0209   | 0.1117     | -3.7220 | 3.0424     | -0.4228 | 0.0068    |
| RMSE                | 0.0129   | 0.1008   | 0.0060   | 0.0089     | 1.6786  | 0.5773     | 0.0612  | 0.0023    |
| High Jump Intensity |          |          |          |            |         |            |         |           |
| True                | 0.0500   | 0.8000   | 0.015    | 0.1000     | -3.0000 | 3.5000     | -0.4000 | 0.0150    |
| Mean                | 0.0537   | 0.8739   | 0.0214   | 0.1169     | -3.3374 | 3.5955     | -0.4129 | 0.0154    |
| RMSE                | 0.0135   | 0.3849   | 0.0060   | 0.0245     | 0.8584  | 0.4732     | 0.0660  | 0.0031    |

Table VIII: This table provides a summary of the simulation results for the SVCJ model. 100 sample paths were simulated off the true parameters for and we report the average across the simulations as well as the root mean squared error (RMSE).

|      | $\alpha$ | $\theta$   | $\kappa$ | $\sigma_v$ | $\rho$    |
|------|----------|------------|----------|------------|-----------|
| True | 0.0500   | 0.5000     | 0.0300   | 0.1000     | -0.5000   |
| Mean | 0.0507   | 0.4755     | 0.0330   | 0.0817     | -0.4921   |
| RMSE | 0.0114   | 0.0601     | 0.0062   | 0.0076     | 0.0711    |
|      | $\mu_y$  | $\sigma_y$ | $\rho_J$ | $\mu_v$    | $\lambda$ |
| True | -2.0000  | 3.5000     | -0.4000  | 1.0000     | 0.0080    |
| Mean | -2.3140  | 2.9116     | -0.0599  | 1.4272     | 0.0093    |
| RMSE | 1.1087   | 0.4259     | 0.4369   | 0.3622     | 0.0027    |

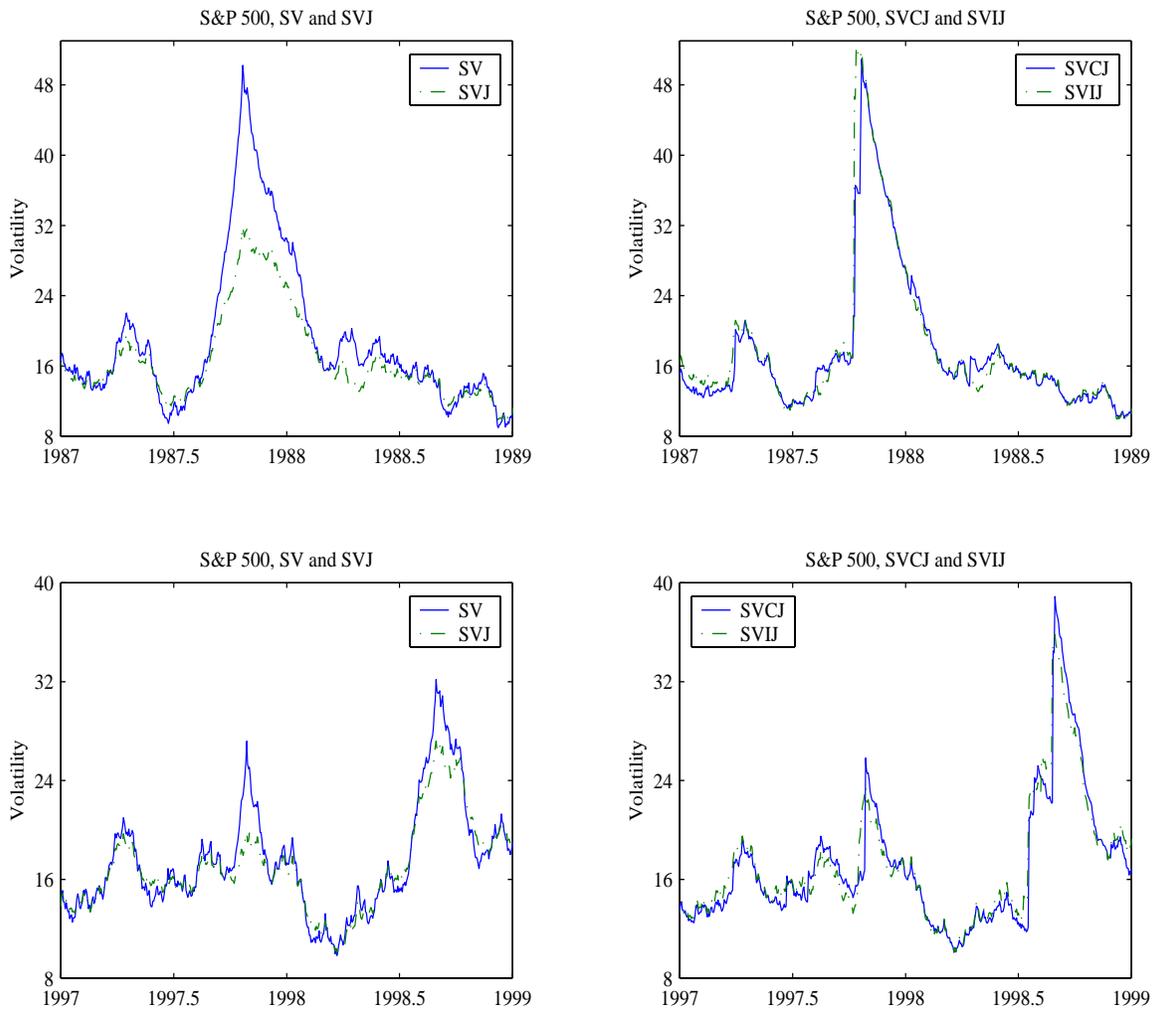


Figure 1: Estimated volatility paths for the S&P 500 index for the four models.

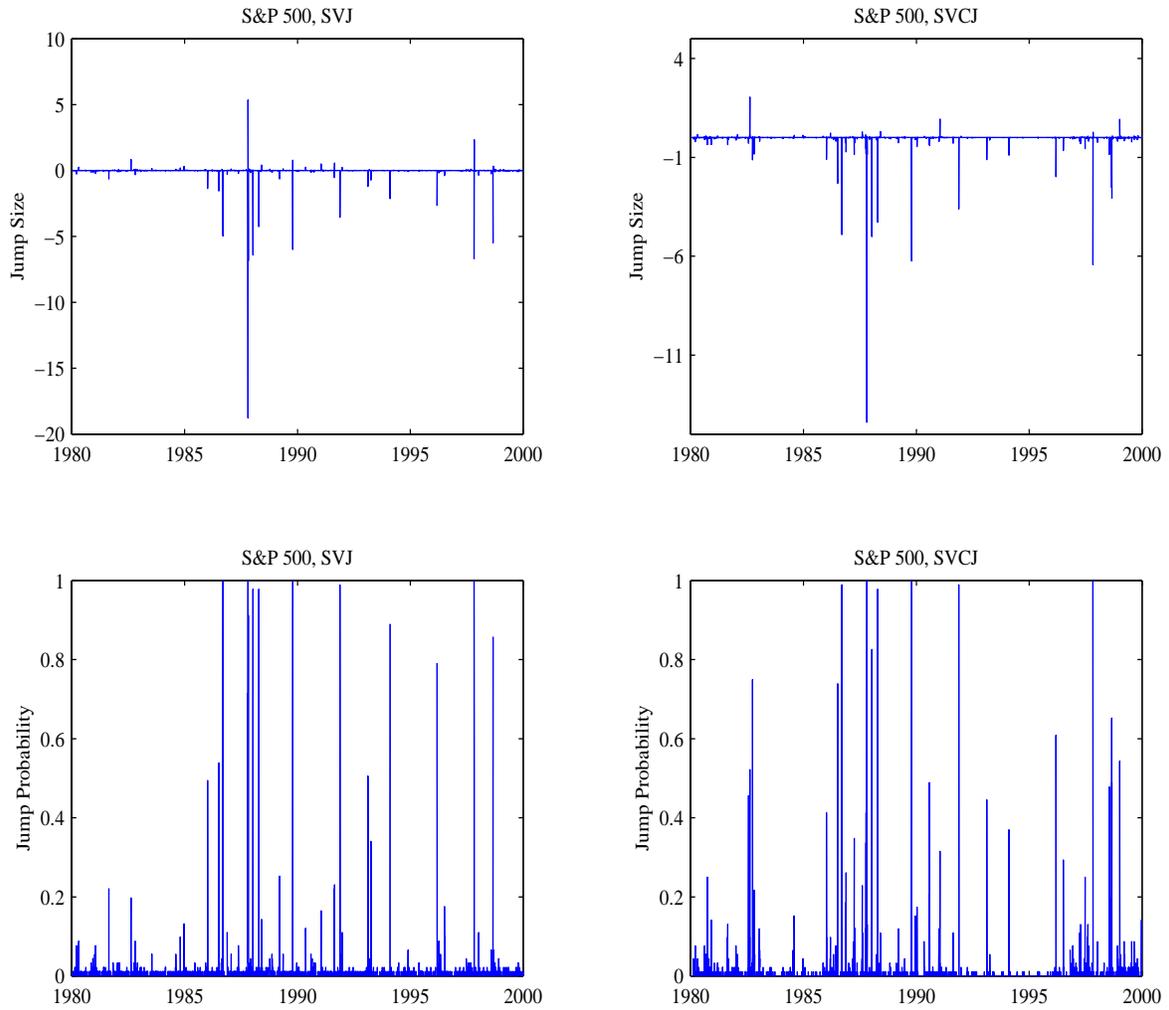


Figure 2: Estimated jump times and sizes for the SVJ and SVCJ models for the S&P 500 index.

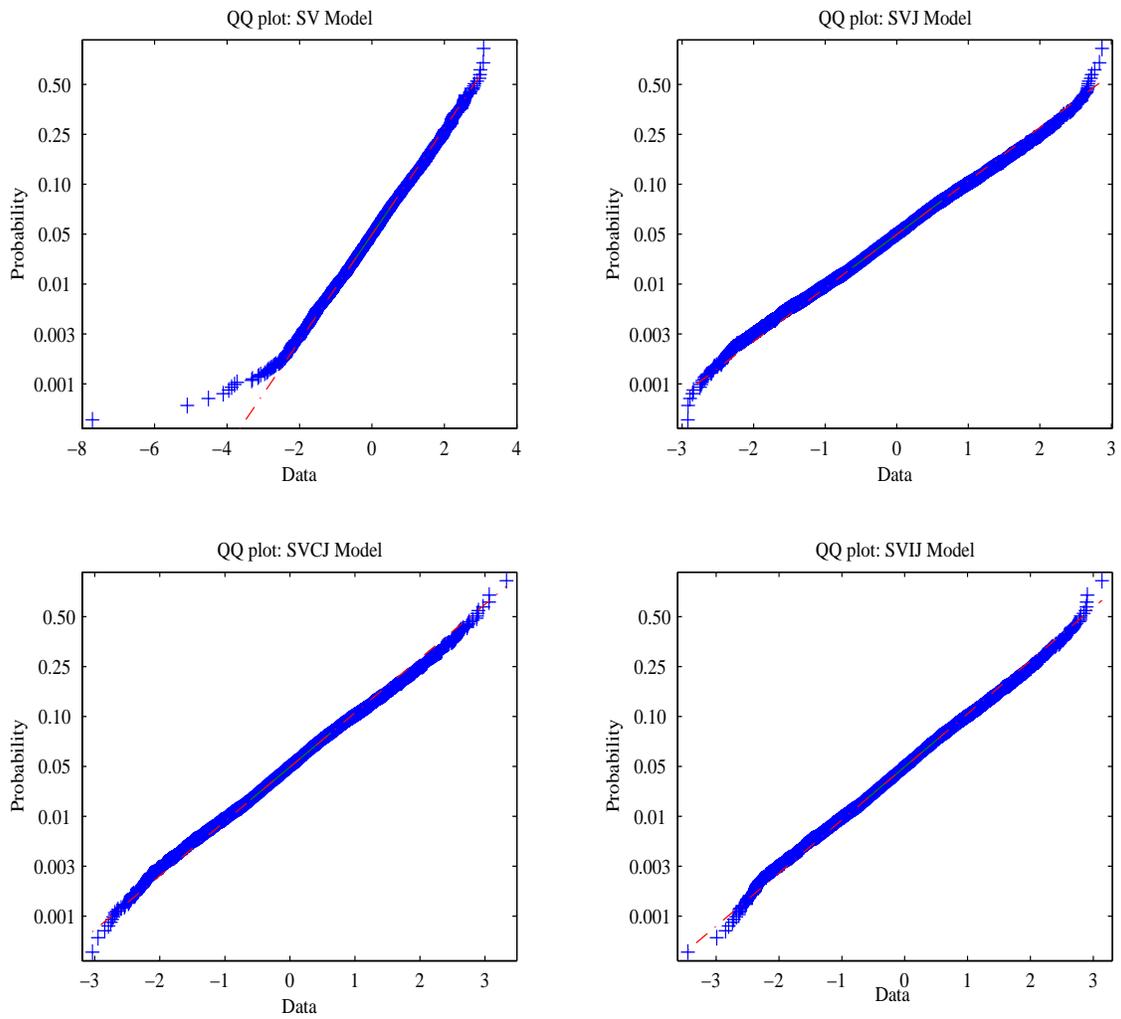


Figure 3: QQ or normality plot of the residuals for each of the models.

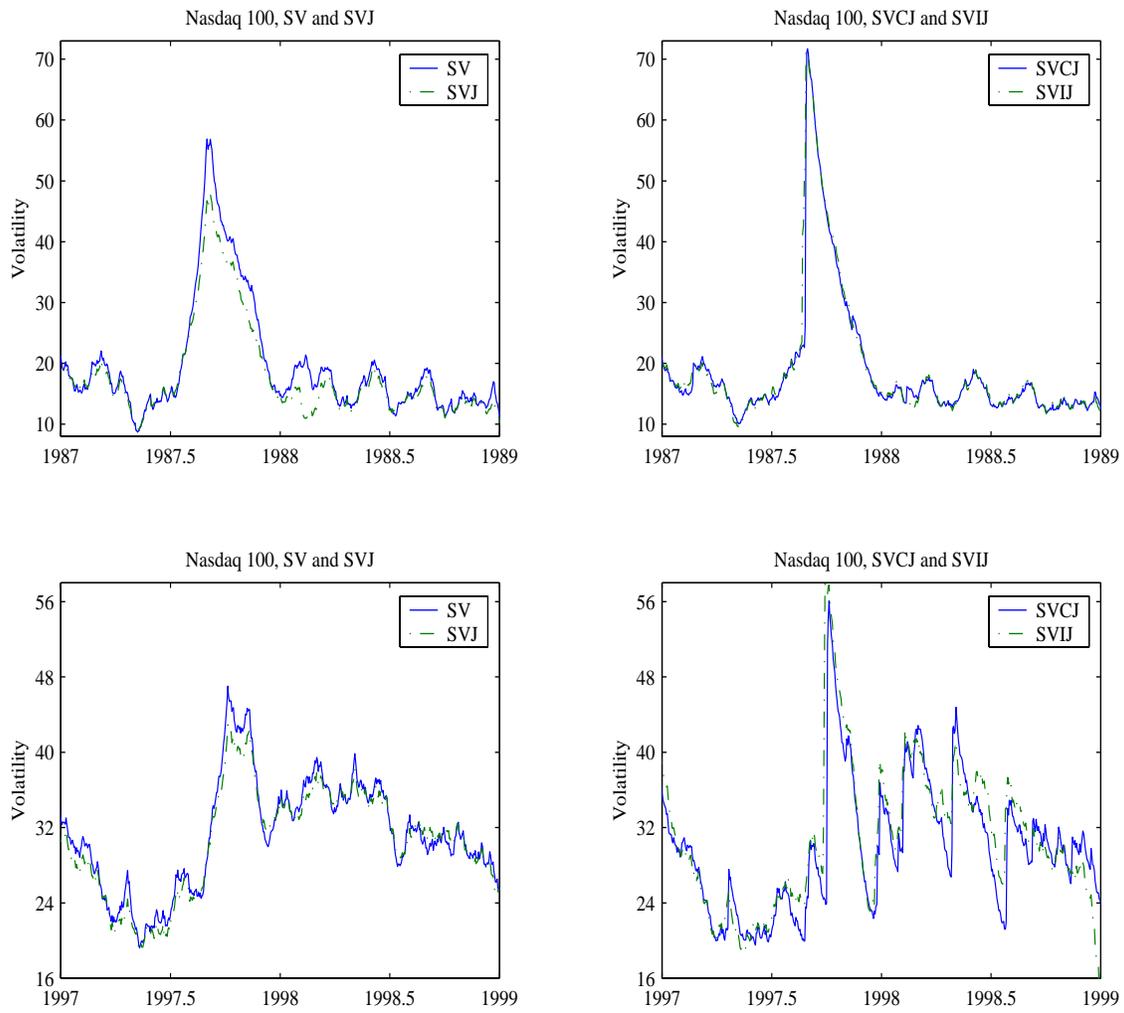


Figure 4: Estimated volatility paths for the Nasdaq 100 for the four models.

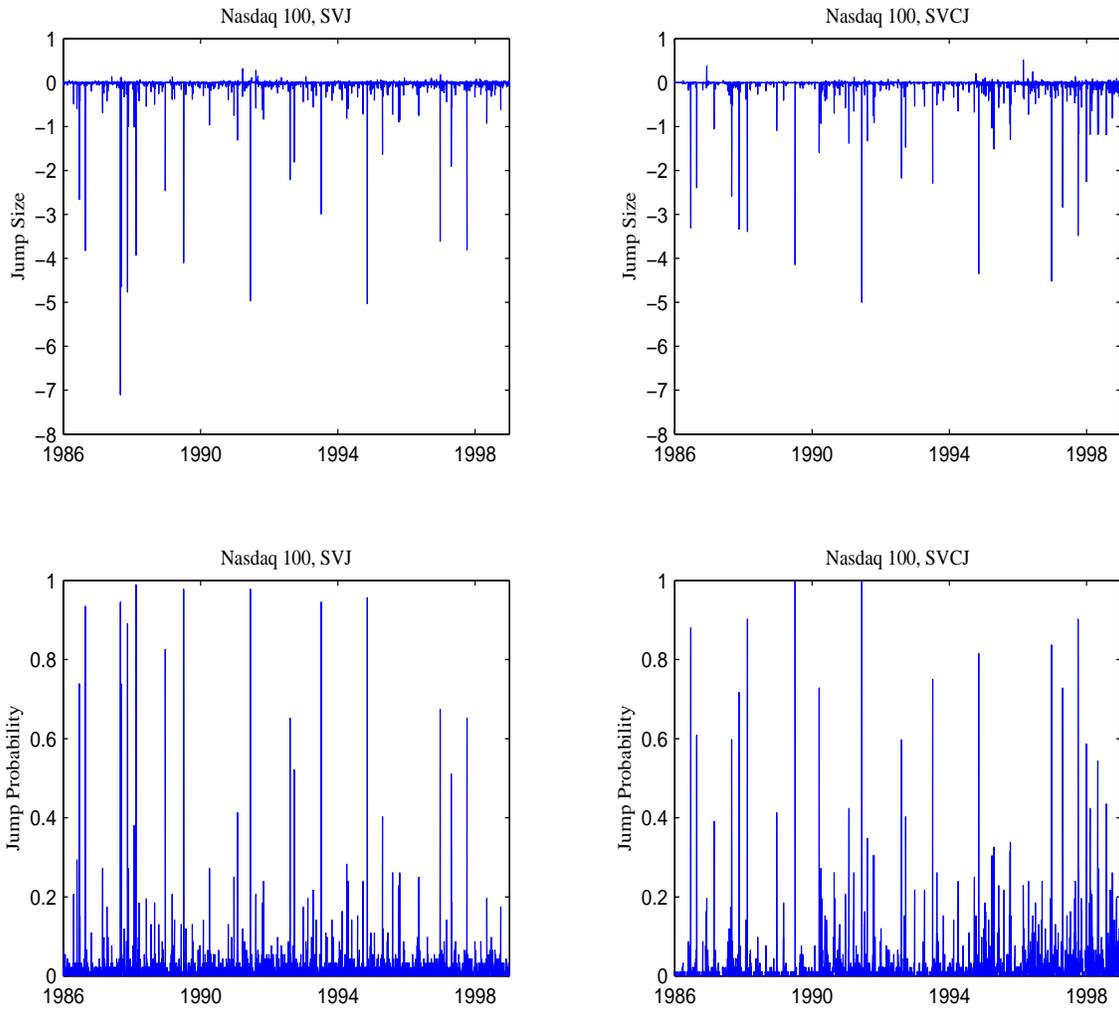


Figure 5: Estimated jump times and sizes for the SVJ and SVCJ models for the Nasdaq 100 index.

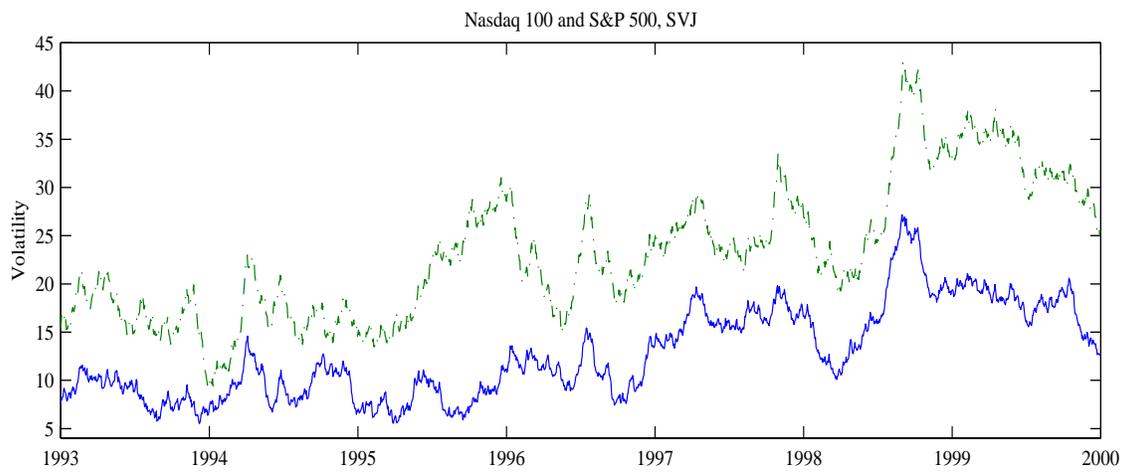
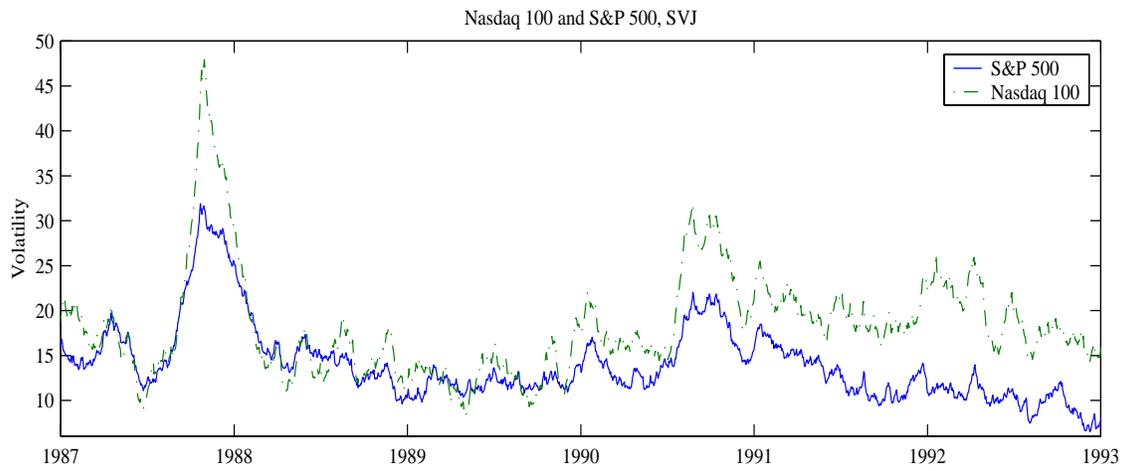


Figure 6: Implied volatility paths for the S&P 500 and Nasdaq 100 for the SVJ model.

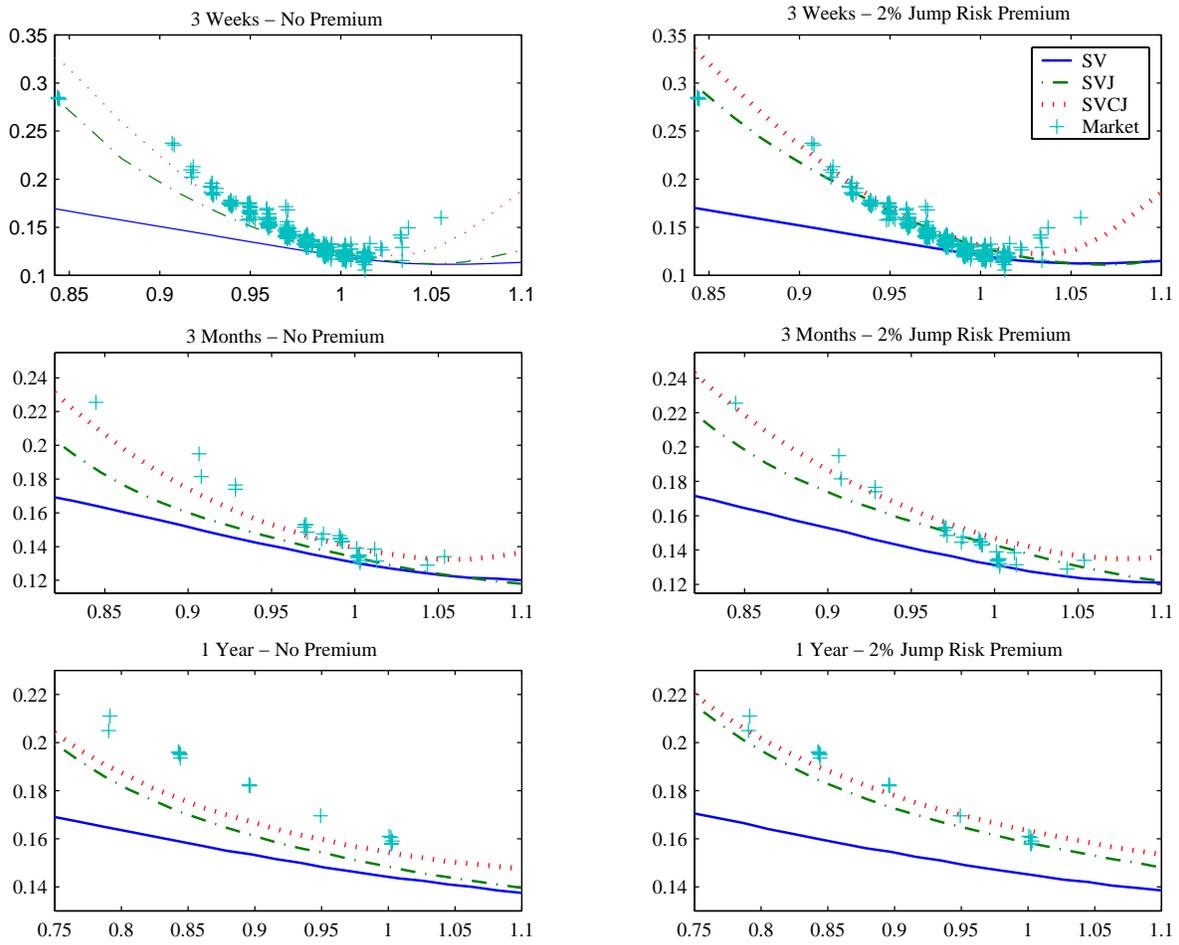


Figure 7: Implied volatility curves for the SV, SVJ and SVCJ models and market implied volatility curves for a randomly selected day in our sample, October 31, 1994.

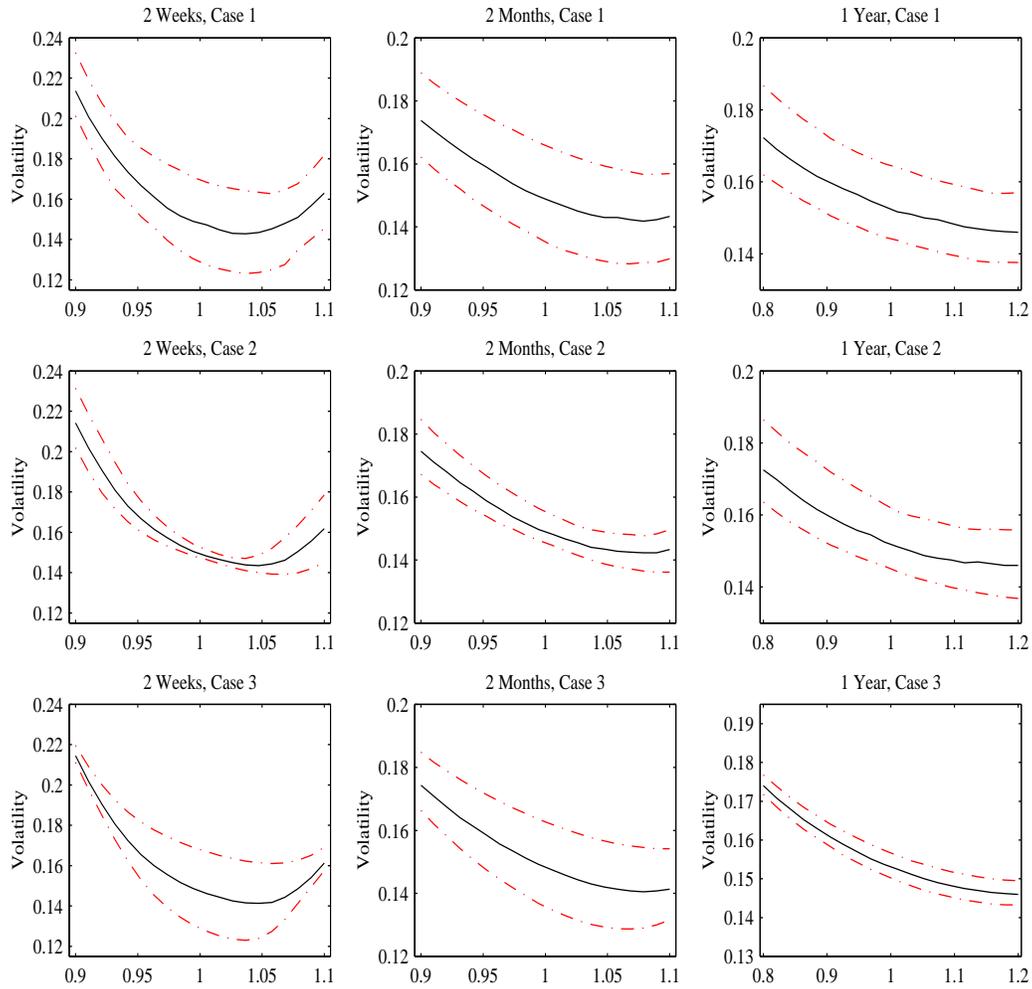


Figure 8: The top panels integrate out all uncertainty. The middle panels integrate out only parameter uncertainty (conditional on estimated volatility). The bottom panels integrate out only volatility uncertainty (conditional on posterior mean of parameters).