Online Appendix

Credit Crises, Precautionary Savings, and the Liquidity Trap

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A.1 Analytical results

Here we derive three analytical results, which are useful to interpret the responses of consumption and output in the baseline model.

The first is a proposition due to Adrien Auclert, which shows the relation between the response of consumption to a tightening credit limit in partial equilibrium and the marginal propensity to consume.

Let $C(b, \theta; \tau, \phi)$ denote the optimal consumption policy in our model in steady state, where we make explicit the dependence of the policy on the lump sum tax τ and on the debt limit ϕ.

Proposition 1 *The consumption function C satisfies:*

$$\frac{\partial C}{\partial \phi} = \frac{\partial C}{\partial b} + (1-q)\frac{\partial C}{\partial \tau}.$$

Proof. Let us first prove the following for any $\Delta \geq 0$

$$C(b, \theta; \phi, \tau) = C(b + \Delta, \theta; \phi - \Delta, \tau + (1-q)\Delta). \tag{6}$$

This result can be proved by arguing that any path for consumption and labor supply that is feasible at $(b, \theta; \phi, \tau)$ is also feasible $(b + \Delta, \theta; \phi - \Delta, \tau + (1-q)\Delta)$, and the reverse also holds. The argument for this statement relies on the fact that the pair c, y satisfies the budget constraint

$$b + y - \tau = c + qb',$$

if and only if it satisfies the budget constraint

$$(b + \Delta) + y - [\tau + (1-q)\Delta] = c + q(b' + \Delta),$$

and that $b \geq -\phi$ iff $b + \Delta \geq -(\phi - \Delta)$ for all b. The proposition follows from differentiating both sides of (6) with respect to Δ and rearraning. ∎

An immediate corollary of this result is that if the interest rate is zero in equilibrium ($q = 1$), the marginal response to a tightening of credit limit $d\phi$ is equivalent to the response to a one time transfer $-d\phi$.

A second analytical result illustrates the relevance of assumptions about curvature of the policy functions for the model's output predictions under flexible prices. The following neutrality result, due to Ivan Werning, shows a special case in which deleveraging is perfectly output-neutral under flexible prices.

Proposition 2 *Suppose there is no unemployment state, i.e., all realizations of θ are positive, and suppose $\gamma = \eta = 1$. Then a shock to the credit limit has no effect on output.*

Proof. Define

$$z_{it} \equiv b_{it} - q_t b_{it+1} - \tau_t,$$

and notice that consumption and labor supply conditional on z_{it} must maximize $U(c_{it}, n_{it})$ subject to $c_{it} = \theta_{it} n_{it} + z_{it}$. Denote the optimum c_{it} of this static problem with the function $c_{it} = \mathcal{C}(z_{it}, \theta_{it})$. Let $H_t(z, \theta)$ denote the joint CDF of z_{it} and θ_{it} in equilibrium. Equilibrium output is then

$$Y_t = \int \mathcal{C}(z, \theta)\, dH_t(z, \theta),$$

and bonds market clearing requires

$$\int z dH_t(z, \theta) = 0.$$

Under log preferences the first order condition of the static problem is

$$\frac{\theta}{c} = \psi \frac{1}{1-n}.$$

Rearranging and using the budget constraint yields

$$\psi c = \theta(1-n) = \theta - (c - z),$$

so we can derive explicitly

$$\mathcal{C}(z, \theta) = \frac{\theta + z}{1 + \psi}.$$

Then in equilibrium we have

$$Y_t = \int \mathcal{C}(z,\theta)\, dH_t(z,\theta) = \frac{\int \theta dH_t(z,\theta) + \int z dH_t(z,\theta)}{1+\psi} = \frac{E[\theta]}{1+\psi},$$

where we use bonds market clearing and the fact that the marginal of θ is given by the productivity process. ■

The third result is a neutrality result which helps clarify why an increase in bond supply can undo the effects of a contraction in the debt limit.

Proposition 3 *An economy with debt limits and bond supplies $\{\phi_t, B_t\}$ is equivalent, in terms of equilibrium outcomes for interest rates and individual consumption and labor supply, to an economy with debt limits and bond supplies $\{\phi'_t, B'_t\}$ if*

$$B'_t + \phi'_t = B_t + \phi_t.$$

Proof. Write the individual budget constraint as

$$q_t b_{it+1} + c_{it} = y_{it} + b_{it} - \tau_t + \iota_{it}\nu_t,$$

where ι_{it} is an indicator variable equal to 1 iff $\theta_{it} = 0$. Suppose that $q_t, b_{it}, c_{it}, y_{it}$ are part of an equilibrium with debt limits and bond supplies $\{\phi_t, B_t\}$. Consider now the economy with the sequence of borrowing limits $\{\phi'_t, B'_t\}$. Then $q_t, b'_{it}, c_{it}, y_{it}$ are part of an equilibrium with

$$b'_{it+1} = b_{it+1} - B_{t+1} + B'_{t+1}.$$

To prove this notice that, subtracting side by side the government budget constraint, the individual budget constraint in the first equilibrium can be written as

$$q_t(b_{it+1} - B_{t+1}) + c_{it} = y_{it} + (b_{it} - B_t) - u\nu_t + \iota_{it}\nu_t,$$

and then the individual budget constraint in the second equilibrium is also satisfied with

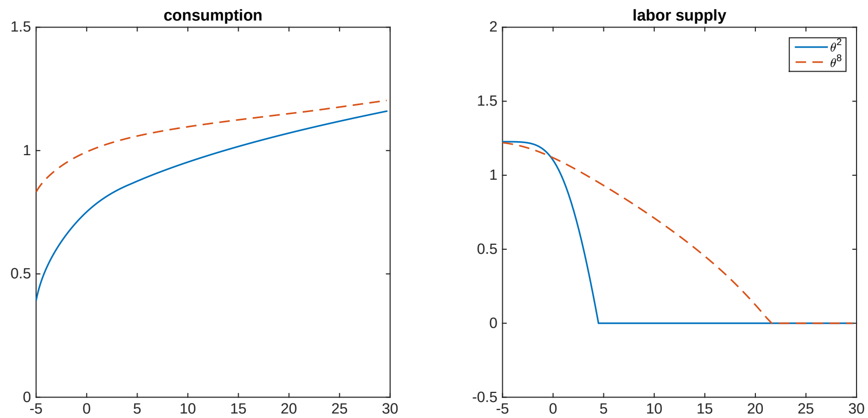
$$q_t\left(b'_{it+1} - B'_{t+1}\right) + c_{it} = y_{it} + \left(b'_{it} - B'_t\right) - u\nu_t + \iota_{it}\nu_t.$$

The optimality conditions are unaffected by these changes and the borrowing constraint is satisfied because

$$b'_{it+1} - B'_{t+1} = b_{it+1} - B_{t+1} \geq -\phi_{t+1} - B_{t+1} = -\phi'_{t+1} - B'_{t+1}.$$

■

Figure 1: *Optimal Consumption and Labor Supply in Steady State: Low ψ Calibration*



A.2 Alternative calibrations

In all calibrations the income process is calibrated as in the baseline. In Table 1 we report the parameters for our first three alternative calibrations. For all three, γ is set at 4. Table 2 reports the parameters for the alternative calibration with $\gamma = 6$. Figure 1 plots consumption and labor supply policies for the low ψ calibration.

A.3 Model with durables

Households preferences are represented by the utility function

$$\mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_{it}, k_{it}, n_{it})\right],$$

where c_{it} is non-durable consumption, the service flow from durables is proportional to the stock of durables k_{it}, and n_{it} is labor effort.

Each period durables depreciate at the rate δ and the household chooses whether to increase or decrease its durable stock. To increase the durable stock or keep it constant, the household spends $k_{it+1} - k_{it}$ plus δk_{it} to cover depreciation. To reduce the durable stock the household faces an additional reselling cost proportional to the capital sold, $\zeta \cdot (k_{it} - k_{it+1})$. The parameter $\zeta > 0$ determines the illiquidity of durable goods. These

Table 1: *Alternative Calibrations*

Median wealth calibration:

Parameter	Explanation	Value	Target/Source
β	Discount factor	0.9615	Interest rate $r = 2.5\%$
η	Curvature of utility from leisure	1.5	Average Frisch elasticity = 1
ψ	Coefficient on leisure in utility	19.03	Average hours worked 0.4 of endowment
ν	Unemployment benefit	0.10	40% of average labor income
B	Bond supply	0.5231	Median liquid assets (SCF)
ϕ	Borrowing limit	0.6031	Total gross debt (Flow of funds)

Low ψ calibration:

Parameter	Explanation	Value	Target/Source
β	Discount factor	0.9755	Interest rate $r = 2.5\%$
η	Curvature of utility from leisure	0.25	Average Frisch elasticity = 1
ψ	Coefficient on leisure in utility	1.389	Average hours worked 0.8 of endowment
ν	Unemployment benefit	0.10	40% of average labor income
B	Bond supply	1.6	Median liquid assets (SCF)
ϕ	Borrowing limit	0.7077	Total gross debt (Flow of funds)

Median wealth, low ψ calibration:

Parameter	Explanation	Value	Target/Source
β	Discount factor	0.9544	Interest rate $r = 2.5\%$
η	Curvature of utility from leisure	0.25	Average Frisch elasticity = 1
ψ	Coefficient on leisure in utility	1.427	Average hours worked 0.8 of endowment
ν	Unemployment benefit	0.10	40% of average labor income
B	Bond supply	0.429	Median liquid assets (SCF)
ϕ	Borrowing limit	0.7184	Total gross debt (Flow of funds)

Table 2: *Alternative Calibrations (continued)*

Calibration with $\gamma = 6$:

Parameter	Explanation	Value	Target/Source
β	Discount factor	0.9756	Interest rate $r = 2.5\%$
η	Curvature of utility from leisure	1.5	Average Frisch elasticity = 1
ψ	Coefficient on leisure in utility	101.22	Average hours worked 0.4 of endowment
ν	Unemployment benefit	0.10	40% of average labor income
B	Bond supply	1.6	Median liquid assets (SCF)
ϕ	Borrowing limit	1.100	Total gross debt (Flow of funds)

assumptions are summarized in the adjustment cost function

$$g\left(k_{it+1}, k_{it}\right) = \begin{cases} k_{it+1} - k_{it} + \delta k_{it} & \text{if } k_{it+1} \geq k_{it} \\ (1-\zeta)\left(k_{it+1} - k_{it}\right) + \delta k_{it} & \text{if } k_{it+1} < k_{it} \end{cases} .$$

We assume that $1 - \zeta > \delta$, so the household can always liquidate part of its durable stock to cover for depreciation.[37]

If a household is a net seller of bonds, i.e., if $b_{it+1} < 0$, the household needs to buy intermediation services from a competitive banking sector. A banking firm incurs a proportional intermediation cost of χ per dollar of bonds issued, which captures monitoring and collection costs. This implies that the household receives a net price $\hat{q}_t = (1-\chi)\, q_t$ per bond issued and banks make zero profits. Letting b^+_{it} denote positive bond holdings and b^-_{it} denote negative holdings, the household's budget constraint is then

$$q_t b^+_{it+1} + \hat{q}_t b^-_{it+1} + g\left(k_{it+1}, k_{it}\right) + c_{it} + \tilde{\tau}_{it} \leq b_{it} + y_{it},$$

where the tax $\tilde{\tau}_{it}$ depends on the household's productivity θ_{it} as in the baseline model.

The production side of the model is as in the benchmark model, with a linear production function $y_{it} = \theta_{it} n_{it}$ and an exogenous Markov process for θ_{it}. Durable and non-

[37] An alternative approach to modeling transaction costs—made in Grossman and Laroque (1990) and Gruber and Martin (2003)—is to assume that when agents choose $k_{t+1} \neq k_t$ they have to sell k_t at price $(1-\zeta)\, k_t$ and buy k_{t+1} at full price. So $g\left(k_{t+1}, k_t\right) = k_{t+1} - (1-\zeta)\, k_t + \delta k_t$ if $k_{t+1} \neq k_t$ and $g\left(k_{t+1}, k_t\right) = \delta k_t$ otherwise. A large literature takes explicitly into account the lumpiness of durable purchases associated to various fixed costs of adjustment (e.g., Caballero, 1990, and more recently Leahy and Zeira, 2005). An advantage of our approach is that it keeps the household's problem concave.

durable goods are produced with the same technology, so the relative price of durables is 1.

The household's borrowing constraint is

$$b_{it+1} \geq -\phi_k k_{it+1}. \tag{7}$$

The household debt is collateralized by its durable holdings. The parameter ϕ_k is the fraction of the value of the durable that can be used as collateral.

The government budget constraint is unchanged:

$$B_t + \nu_t u = q_t B_{t+1} + \tau_t.$$

As in the baseline, we fix the supply of government bonds and the unemployment benefit at the levels B and ν, and let the tax τ_t adjust to satisfy budget balance.

A.3.1 Equilibrium and calibration

The main difference with the baseline model is that durable goods are now an additional state variable. Optimal household policies are now functions of the three-dimensional state (b, k, θ): the initial stock of bonds, the initial stock of durables, and current productivity. These three states fully determine the household's choice of non-durable and durable purchases, labor supply and the optimal level of borrowing or lending.

Let $\Psi_t(b, k, \theta)$ denote the joint distribution of b, k and θ in the population. Combining the household's optimal transition for bond holdings and durable goods with the exogenous Markov process for productivity, we obtain the transition probability of the individual state, and, aggregating, a transition for the distribution Ψ_t. The definition of equilibrium is then the natural generalization of Definition 1, where the bonds market clearing condition is now

$$\int b d\Psi_t(b, k, \theta) = B.$$

To calibrate the model we adopt the utility function:

$$U(c, k, n) = \frac{\left(c^{\alpha} k^{1-\alpha}\right)^{1-\gamma}}{1-\gamma} + \psi \frac{(1-n)^{1-\eta}}{1-\eta}.$$

Table 3: *Parameter Values: Durable Model*

Parameter	Explanation	Value	Target/Source
β	Discount factor	0.9713	Interest rate $r = 2.5\%$
γ	Coefficient of relative risk aversion	4	
η	Curvature of utility from leisure	1.50	Average Frisch elasticity $= 1$
ψ	Coefficient on leisure in utility	2.54	Average hours worked $= 0.4$
α	Coefficient on non-durables	0.7	Ratio of non-durable and non-housing services to total personal consumption expenditures in NIPA (2000-10 average)
δ	Durables depreciation rate	0.0129	BEA Fixed Asset Tables ratio of depreciation to net stock, (2000-8 average, Hall, 2011b)
ζ	Proportional loss on durable sales	0.15	
χ	Intermediation cost	0.01	
ρ	Persistence of productivity shock	0.967	Persistence of wage process in Floden and Lindé (2001)
σ_ε	Variance of productivity shock	0.017	Variance of wage process in Floden and Lindé (2001)
$\pi_{e,u}$	Transition to unemployment	0.057	Shimer (2005)
$\pi_{u,e}$	Transition to employment	0.882	Shimer (2005)
ν	Unemployment benefit	0.160	40% of average labor income (Shimer, 2005)
ϕ_k	Max loan-to-value ratio	0.8	
B	Bond supply	1.60	Liquid-assets-to-GDP ratio of 1.78

Note: The quantities ν and B are expressed in terms of yearly aggregate output. See the text for details on the targets.

We choose a simple Cobb-Douglas specification to aggregate durable and non-durable consumption.[38] Therefore, α is the ratio of non-durable consumption to total consumption.[39] To compute this ratio we compute durables as the sum of durable consumption and consumption of housing services from NIPA. We take all other consumption (non-durable goods and non-housing services) as nondurables. The average value for 2000-2010 gives us $\alpha = 0.7$. As in our baseline exercise, we choose β to get a 2.5% yearly interest rate, set the coefficient of risk aversion $\gamma = 4$, choose η to obtain an average Frisch elasticity of 1, and choose ψ so that average hours worked are 40% of the time endowment. Also for the wage process, the transitions between employment and unemployment, and the unemployment benefit we follow our baseline calibration.[40]

For the accumulation of durable goods, we need to choose δ and ζ. We set $\delta = 1.29\%$ to match the depreciation rate from NIPA Fixed Assets Tables. The parameter ζ represents the cost of selling durable goods and captures their illiquidity. We set it to 15%.

Finally we need to choose ϕ_k, the intermediation cost χ, and the bond supply B. We set ϕ_k to 0.8, which is in the range of loan-to-value ratios in mortgages and durable loans.[41] The parameter χ is set at 1%, so our exercise starts from a fairly low initial spread. The supply of government bonds B is chosen as in the baseline, to match the ratio of liquid assets to GDP equal to 1.78. The parameters used are summarized in Table 2.

A.3.2 Characterization and steady state

The new ingredient, relative to the baseline problem, is that households face a portfolio choice. Each period, households choose their labor effort n_{it} and non-durable consumption c_{it} as in the baseline model. These choices determine their saving, gross of durable purchases, $y_{it} - c_{it} - \tilde{\tau}_{it}$ (from now on, "gross saving"). Households then need to choose

[38] Ogaki and Reinhart (1998) offer evidence in favor of an elasticity of substitution between durables and non-durables close to 1.

[39] Due to the presence of the collateral constraint, this relation is not exact (as the cost of durables is reduced by the shadow price of the collateral constraint), but it approximately holds in our simulations.

[40] However, we now approximate the wage with a 5-state Markov chain, for computational reasons.

[41] The debt-to-GDP ratio we obtain is 54%. We do not try to choose ϕ_k to match observed debt-to-GDP ratios in the household sector (which prior to the crisis went above 100%) because, given our other parameters, our model cannot deliver debt-to-GDP ratios above 70%.

how to allocate this saving between durable purchases and bond accumulation. The optimality conditions characterizing household behavior are derived in the Online Appendix A.3. Here we provide some intuition for the optimal portfolio dynamics.

The kinked adjustment cost for durables implies that the optimal portfolio is characterized by two adjustment bands. In particular, for a given productivity θ_{it}, the optimal portfolio (b_{it+1}, k_{it+1}) always lies in a region like the grey region in Figure 2 (which corresponds to $\theta_{it} = 1$). The dashed green line corresponds to the borrowing limit, which is proportional to durable holdings. Given an initial capital stock, say $k_{it} = 7$, the locus of possible optimal portfolios is given by the solid blue line.[42] Our numerical analysis shows that a household starting at portfolio P_1 ($b_{it} = -1.5, k_{it} = 7$) will choose positive gross saving, keep its durable holdings unchanged, and allocate its gross saving to debt repayment, moving along the arrow originating at P_1. A more indebted household, starting at portfolio P_2 ($b_{it} = -5.3, k_{it} = 7$) will also choose positive gross saving, but it will also sell some durables and use its gross saving plus the receipts from the durables sale to repay its debt, moving along the arrow originating at P_2 so as to reach the blue line. The common element is that all households starting at k_{it}, irrespective of their initial bond holdings, will choose an end-of-period portfolio on the blue line.[43]

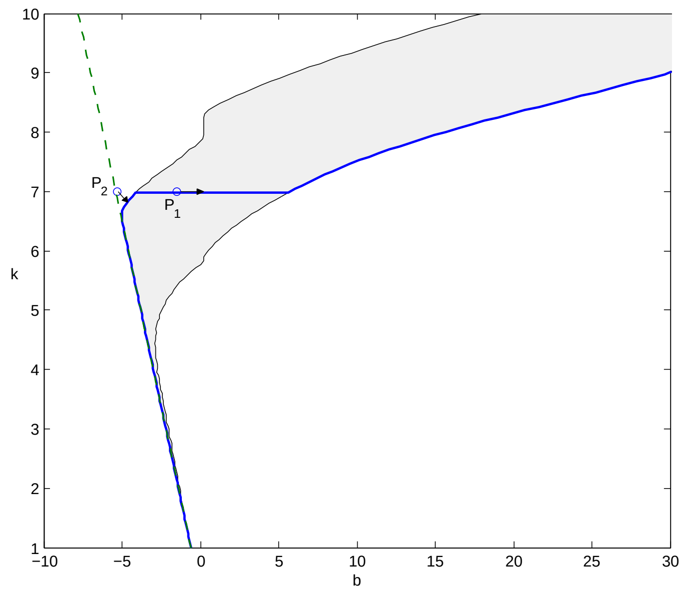
The shape of the adjustment region in Figure 2 is similar for all values of θ. Therefore, the support of the steady state distribution of bonds and durable holdings takes a similar shape, as can be seen in Figure 3. In this figure, we plot the contours of the steady state distribution. The dashed green line represents again the borrowing limit. At low levels of total wealth (bonds plus durables) we find households who hold small durable stocks and small amounts of debt. If a household receives a positive productivity shock, it responds by accumulating durables and taking on more debt, given that the shock is expected to persist. If the household stays at high productivity, it eventually starts to pay off its debt and then goes on to accumulate positive bond holdings. If instead the household is hit by a sequence of negative shocks, in a first phase it will adjust only by selling bonds and, in a second phase, it will adjusts also by selling durables.[44]

[42]See the appendix for the formal definition of this locus.

[43]Notice that both boundaries of the adjustment region have a vertical segment at $b = 0$. This vertical segment is due to the spread between borrowing and lending rates.

[44]Depending on size of the shock and on initial bond holdings, the first phase can be absent and the household can start selling durables right away.

Figure 2: *Portfolio Choice in the Model with Durables*



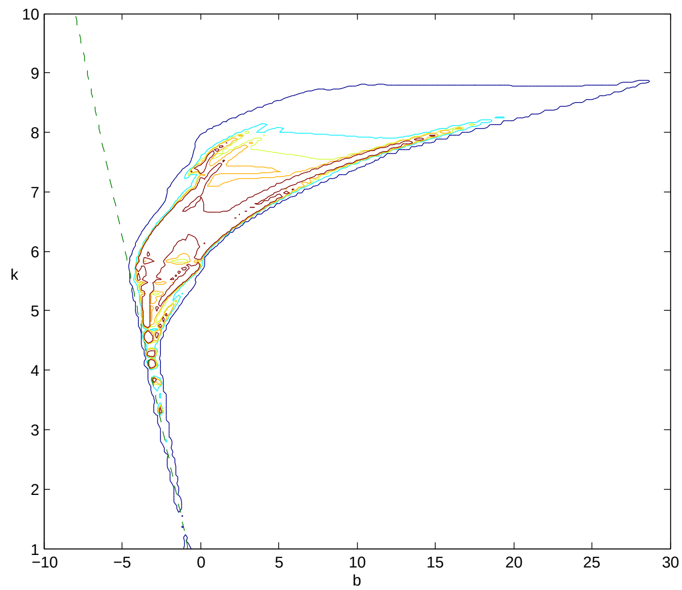
The portfolio dynamics just described help to account for the fact that the distribution tends to be concentrated at the boundaries of the adjustment regions, given that if households are on the boundary and are not hit by a shock, they remain on the boundary. Moreover, there is a mass of households at, or near, the borrowing limit. Unlike in the baseline model, they are not only the households with the lowest total wealth (bonds plus durable holdings), but also middle-wealth households with levered holdings of durable goods.

It is useful to remark two differences between our baseline model and our model with durables. First, the two calibrations lead to very different values for household total net worth. In the baseline, households only hold liquid wealth, and net worth over GDP is 1.60.[45] In the model with durables, households also hold durable wealth, and net worth over GDP is 5.27.[46] However, net worth to GDP is not the only variable determining how important are liquidity constraints, given the different liquidity of the two assets. This point is related to Kaplan and Violante (2011), who also emphasize that

[45]Liquid wealth to GDP is 1.78 and debt to GDP is 0.18.

[46]Liquid wealth to GDP is 1.78, durable wealth over GDP is 4.03, and debt to GDP is 0.54.

Figure 3: *Steady State Distribution of Bonds and Durables*

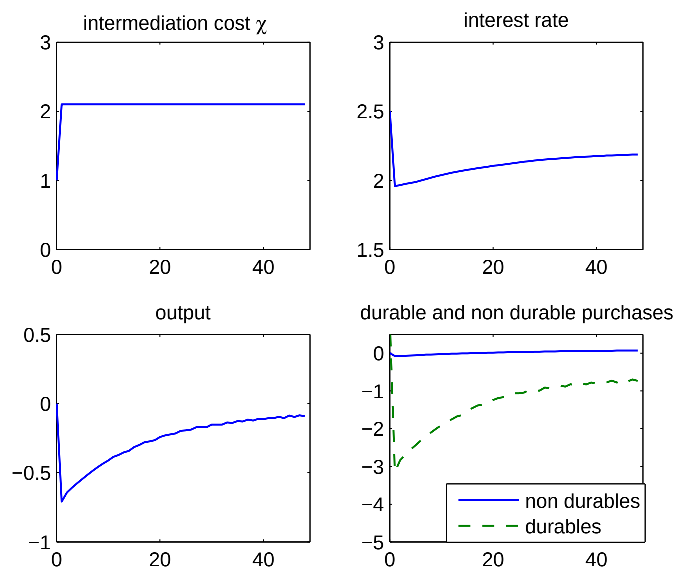


Note: The contour lines correspond to the 0.1%, 10%, 20%, 30%, and 50% percentiles.

adjustment costs imply that "rich" households' consumption behavior can still be far from permanent income predictions.

Second, as argued above, in the baseline model the most indebted households are the households with lowest total wealth, while here they are intermediate-wealth households, with large levered positions in durables. These households can still be induced to adjust nondurable consumption if they are close to their constraint, as seen from the nondurable response in our first exercise (Figure 15). However, when we look at a spread shock that hits all indebted households equally, the typical household hit by the shock now prefers to smooth nondurable consumption and adjust to the shock by selling durables. This helps to explain why the nondurable response is muted in our second and third exercises (Figures 4 and 16) while there is a larger adjustment in durables. In future work, it will be interesting to explore further different combinations of shocks to loan-to-value ratios and to spreads, to understand how they affect differentially households with different initial portfolios, and to compare these results with empirical evidence on the disaggregated response of consumption.

Figure 4: *Responses to a Permanent Shock to the Intermediation Cost χ*



Note: Interest rate is in annual terms. Quantities are in percent deviation from initial steady state.

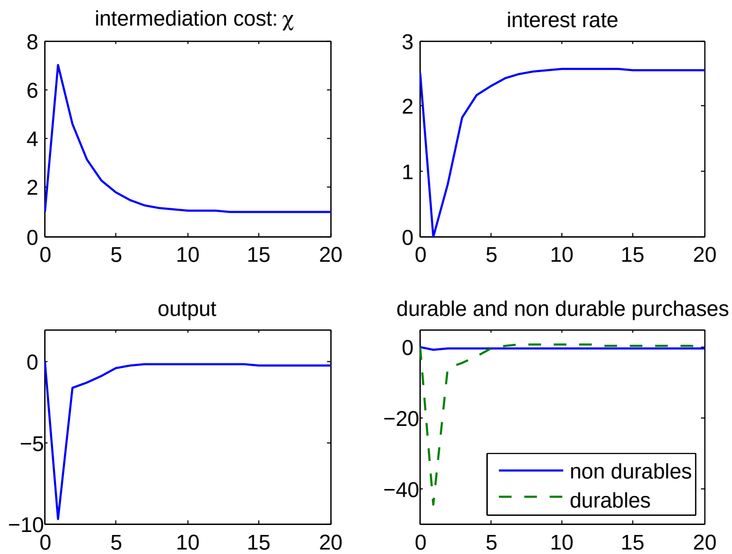
A.3.3 Additional experiments

Here we report additional simulations on the effects of a credit contraction.

First, we show the effects of a permanent spread shock that takes the intermediation cost χ from 1% to 2.21%. As in previous experiments, the size of the shock is chosen to obtain a 10 percentage point long run reduction in the debt-to-GDP ratio. The results are presented in Figure 4. The effects of this shock are much more gradual, as there is no forced deleveraging and borrowers are allowed to adjust their borrowing positions over time. As in the temporary shock described in the text, the shock affects all borrowers and not just those near the borrowing limit. Therefore, the effect is a drop in output, with a contraction in durable purchases of about 3% and an almost negligible drop in non-durables.

In Figure 5 we show the response of the economy to a temporary shock to the intermediation cost in the case of nominal rigidities. The temporary shock to χ is as in the main body of the paper. Now there is a very large contraction in durable purchases

Figure 5: *Responses to a Temporary Shock to the Intermediation Cost χ, with Fixed Prices*



Note: Interest rate is in annual terms. Quantities are in percent deviation from initial steady state.

(-44.5%), which leads to a 9.7% overall output contraction. Once more, we see the effects of the very large interest elasticity of durable purchases. Nominal rigidities cause the real interest rate to be off by about 1.5%, relative to the flexible price case, and this is sufficient to reduce durable purchases by an additional 27%, causing a much deeper recession.

A.4 Computations: baseline model

Here we describe the algorithm used to compute steady states and transitional dynamics of the baseline model. The MATLAB codes are available on our web pages.

Let us begin from the steady state computations. First, we describe how optimal policies and the bond distribution are computed for a given steady state interest rate r. To compute the policy functions $C(b,\theta)$ and $N(b,\theta)$, we iterate on the Euler equation and the optimality condition for labor supply on a discrete grid for the state variable b. To iterate on the policy functions, we use the endogenous gridpoints method of Carroll

(2006). To compute the invariant distribution $\Psi(b,\theta)$ we derive the inverse of the bond accumulation policy, denoted by $g(b,\theta)$, from the policy functions, and update the conditional bond distribution using the formula $\Psi_{(k)}(b|\theta) = \sum_{\tilde{\theta}} \Psi_{(k-1)}(g(b,\tilde{\theta})|\tilde{\theta})P(\tilde{\theta}|\theta)$ for all $b \geq -\phi$, where k stands for the k-th iteration and $P(\tilde{\theta}|\theta)$ is the probability of $\theta_{t-1} = \tilde{\theta}$ conditional on $\theta_t = \theta$. Due to the borrowing constraint, the bond accumulation policy is not invertible at $b = -\phi$, but the formula above still holds defining $g(-\phi,\theta)$ as the largest b such that $b' = -\phi$ is optimal. Finally, we search for the interest rate r that clears the bond market.

To compute transitional dynamics, we get the initial bond distribution $\Psi_0(b,\theta)$ from the initial steady state. We then compute the final steady at $\phi = \phi'$. We choose T large enough that the economy is approximately at the new steady state at $t = T$ (we use $T = 200$ in the simulations reported). Next, we guess a path of interest rates $\{r_t\}$ with $r_T = r'$. We take the consumption policy to be at the final steady state level at $t = T$, setting $C_T(b,\theta) = C'(b,\theta)$, and we compute the sequence of policies $\{C_t(b,\theta), N_t(b,\theta)\}$ using the Euler equation and the optimality condition for labor supply, going backward from $t = T-1$ to $t = 0$ (using the endogenous gridpoints method). Next, we compute the sequence of distributions $\Psi_t(b,\theta)$ going forward from $t = 0$ to $t = T$, starting at $\Psi_0(b,\theta)$, using the optimal policies $\{C_t(b,\theta), N_t(b,\theta)\}$ to derive the bond accumulation policy (using the same updating formula as in the steady state). We then compute the aggregate bond demand B_t for $t = 0, ...T$ and update the interest rate path using the simple linear updating rule $r_t^{(k)} = r_t^{(k-1)} - \epsilon(B_t^{(k)} - \bar{B})$. Choosing the parameter $\epsilon > 0$ small enough the algorithm converges to bond market clearing for all $t = 0, ...T$.

A.5 Computations: model with durables

A.5.1 Derivations

Here we derive optimality conditions for the model with durables. We focus on the steady state for ease of notation, but analogous derivations apply to the transitional dynamics (adding time subscripts). The Bellman equation is

$$\begin{aligned}
V(b,k,\theta) = \max_{c,n,k',b'} \quad & U(c,k,n) + \beta E\left[V\left(b',k',\theta'\right)|\theta\right] \\
s.t. \quad & b + \theta n - \tau(\theta) \geq q b'_+ + \hat{q} b'_- + g\left(k',k\right) + c, \\
& b' + \phi_k k' \geq 0.
\end{aligned}$$

The first order conditions for this problem are as follows. For c and n:

$$\begin{aligned}
U_c(c,k,n) &= \lambda, \\
-U_n(c,k,n) &= \theta\lambda \text{ and } n > 0 \text{ or } -U_n(c,k,n) \geq \theta\lambda \text{ and } n = 0.
\end{aligned}$$

For b' and k':

$$\begin{aligned}
& \beta E\left[V_b\left(b',k',\theta'\right)|\theta\right] + \mu = q\lambda \text{ if } b' > 0, \\
& \beta E\left[V_b\left(b',k',\theta'\right)|\theta\right] + \mu = \hat{q}\lambda \text{ if } b' < 0, \\
q\lambda \geq \; & \beta E\left[V_b\left(b',k',\theta'\right)|\theta\right] + \mu \geq \hat{q}\lambda \text{ if } b' = 0 \\
& \beta E\left[V_k\left(b',k',\theta'\right)|\theta\right] + \mu\phi_k = \lambda \text{ if } k' > k, \\
& \beta E\left[V_k\left(b',k',\theta'\right)|\theta\right] + \mu\phi_k = \lambda(1-\zeta) \text{ if } k' < k, \\
\lambda \geq \; & \beta E\left[V_k\left(b',k',\theta'\right)|\theta\right] + \mu\phi_k \geq \lambda(1-\zeta) \text{ if } k' = k.
\end{aligned}$$

The complementary slackness condition for the borrowing constraint requires that $b' + \phi_k k' = 0$ if $\mu > 0$ and $\mu = 0$ if $b' + \phi_k k' > 0$. The locus of optimal portfolios depicted in Figure 2 corresponds to the set of pairs (b',k') that satisfy the optimality conditions for b' and k' and the complementary slackness condition for μ, for some positive λ.

The envelope conditions are as follows. For b,

$$V_b(b,k,\theta) = \lambda;$$

for k,

$$\begin{aligned}
V_k(b,k,\theta) &= U_k(c,k,n) + \lambda(1-\delta) \text{ if } k' > k \\
V_k(b,k,\theta) &= U_k(c,k,n) + \lambda(1-\delta-\zeta) \text{ if } k' < k
\end{aligned}$$

and

$$V_k(b,k,\theta) = U_k(c,k,n) - \delta\lambda + \beta E\left[V_k\left(b',k',\theta'\right)|\theta\right] + \mu\phi_k \text{ if } k' = k.$$

Using the first order conditions, the envelope condition for k can be written compactly as

$$V_k(b,k,\theta) = U_k(c,k,n) - \delta\lambda + \beta E\left[V_k(b',k',\theta')\,|\theta\right] + \mu\phi_k.$$

A.5.2 Algorithm

Here we describe the algorithm used to compute the model with durables. The MATLAB codes are available on our web pages.

The computation of the model with durables also exploits the endogenous gridpoints method. However, adapting this method to the case of two endogenous state variables requires some extra steps, which are described here. Our approach is similar to Hintermaier and Koeniger (2010), in that we first find the subset of potentially optimal portfolios in the space (b',k') and then take the backward step typical of the endogenous gridpoints method only starting from pairs (b',k') in this subspace. However, unlike Hintermaier and Koeniger (2010), our approach focuses on computing the partial derivatives of the value function instead that on the policy functions.

Define

$$\mathrm{V}_b(b',k',\theta) \equiv E\left[V_b(b',k',\theta')\,|\theta\right], \tag{8}$$

$$\mathrm{V}_k(b',k',\theta) \equiv E\left[V_k(b',k',\theta')\,|\theta\right]. \tag{9}$$

Our objective is to approximate the functions V_b and V_k with piecewise linear functions on the discrete grids $\{b^0,...,b^n\}$ and $\{k^0,...,k^m\}$. We start with an initial guess for V_b and V_k and proceed as follows.

1. *Find the set of pairs (b',k') that are optimal for some state (b,k,θ).* To do so, let k and k' vary (independently) on the grid $\{k^0,...,k^m\}$ and let θ vary on $\{\theta^0,...,\theta^S\}$. For each tripe (k,k',θ), three cases are possible: $k' > k$, $k' < k$ and $k' = k$. In each case, we want to find the value(s) of b' consistent with optimality.

   (a) If $k' > k$, choose the value of b' that satisfies one of the following optimality conditions:

$$\begin{aligned} q\mathrm{V}_k\left(b',k',\theta\right) &= \mathrm{V}_b\left(b',k',\theta\right) \text{ and } b'>0, \\ \text{or } q\mathrm{V}_k\left(b',k',\theta\right) &\geq \mathrm{V}_b\left(b',k',\theta\right) \geq \hat{q}\mathrm{V}_k\left(b',k',\theta\right) \text{ and } b'=0, \\ \text{or } \hat{q}\mathrm{V}_k\left(b',k',\theta\right) &= \mathrm{V}_b\left(b',k',\theta\right) \text{ and } -\phi_k k' < b' < 0, \\ \text{or } \hat{q}\mathrm{V}_k\left(b',k',\theta\right) &\geq \mathrm{V}_b\left(b',k',\theta\right) \text{ and } b' = -\phi_k k'. \end{aligned}$$

(b) If $k' < k$, choose the value of b' that satisfies one of the following optimality conditions:

$$\begin{aligned} q\mathrm{V}_k\left(b',k',\theta\right) &= (1-\zeta)\,\mathrm{V}_b\left(b',k',\theta\right) \text{ and } b'>0, \\ \text{or } q\mathrm{V}_k\left(b',k',\theta\right) &\geq (1-\zeta)\,\mathrm{V}_b\left(b',k',\theta\right) \geq \hat{q}\mathrm{V}_k\left(b',k',\theta\right) \text{ and } b'=0, \\ \text{or } \hat{q}\mathrm{V}_k\left(b',k',\theta\right) &= (1-\zeta)\,\mathrm{V}_b\left(b',k',\theta\right) \text{ and } -\phi_k k' < b' < 0, \\ \text{or } \hat{q}\mathrm{V}_k\left(b',k',\theta\right) &\geq (1-\zeta)\,\mathrm{V}_b\left(b',k',\theta\right) \text{ and } b' = -\phi_k k'. \end{aligned}$$

(c) If $k' = k$, there is an interval of values of b' consistent with optimality, which we denote $[b'_L, b'_U]$. b'_L is the value that solves the conditions in (1.a) and b'_U is the value that solves the conditions in (1.b). Clearly, in some cases $b'_L = b'_U$ and the interval is degenerate.

2. *Derive the associated values of the Lagrange multipliers.* For each tripe (k, k', θ), given the value(s) of b' found in step 1, we derive values for the Lagrange multipliers λ and μ. Again, there are three cases.

   (a) $k' > k$. If the associated b' is equal to $-\phi_k k'$ find μ that solves

$$\beta\mathrm{V}_b\left(b',k',\theta\right) + \mu = \beta\hat{q}\mathrm{V}_k\left(b',k',\theta\right) + \phi_k\hat{q}\mu,$$

   otherwise set $\mu = 0$. Then set $\lambda = \beta\mathrm{V}_k\left(b',k',\theta\right) + \phi_k\mu$.

   (b) $k' < k$. If the associated b' is equal to $-\phi_k k'$ find μ that solves

$$\beta\left(1-\zeta\right)\mathrm{V}_b\left(b',k',\theta\right) + \left(1-\zeta\right)\mu = \beta\hat{q}\mathrm{V}_k\left(b',k',\theta\right) + \phi_k\hat{q}\mu,$$

   otherwise set $\mu = 0$. Then set $\lambda = \beta\mathrm{V}_k\left(b',k',\theta\right) + \phi_k\mu$.

(c) $k' = k$. Now there are in general different triples (b', μ, λ) consistent with optimality. We derive them as follows, depending on the values b'_L and b'_U derived in (1.c).

i. If $b'_L = b'_U = -\phi_k k'$, form a sequence of triples (b', μ, λ) with $b' = -\phi_k k'$, μ taking values in the interval

$$\left[\beta \frac{\hat{q}V_k(b', k', \theta) - V_b(b', k', \theta)}{1 - \phi_k \hat{q}}, \beta \frac{\hat{q}V_k(b', k', \theta) - (1 - \zeta) V_b(b', k', \theta)}{1 - \zeta - \phi_k \hat{q}}\right],$$

and $\lambda = (\beta V_b(b', k', \theta) + \mu) / \hat{q}$.

ii. If $b'_L = -\phi_k k' < b'_U$, form a sequence of triples (b', μ, λ) as follows: first, a sequence of triples with $b' = -\phi - \phi_k k'$, μ taking values in the interval

$$\left[0, \beta \frac{\hat{q}V_k(b', k', \theta) - (1 - \zeta) V_b(b', k', \theta)}{1 - \zeta - \phi_k \hat{q}}\right],$$

and $\lambda = (\beta V_b(b', k', \theta) + \mu) / \hat{q}$; next, a sequence with b' taking values in the interval $(b'_L, b'_U]$, $\mu = 0$, and $\lambda = \beta V_b(b', k', \theta) / \hat{q}$ if $b' < 0$ and $\lambda = \beta V_b(b', k', \theta) / q$ if $b' > 0$.

iii. If $-\phi_k k' < b'_L < b'_U$, form a sequence of triples (b', μ, λ) with b' taking values in $[b'_L, b'_U]$, $\mu = 0$, and $\lambda = \beta V_b(b', k', \theta) / \hat{q}$ if $b' < 0$ and $\lambda = \beta V_b(b', k', \theta) / q$ if $b' > 0$.

iv. Finally, if $b'_U \geq 0 \geq b'_L$, add to the sequences of triples (b', μ, λ) in (i)-(iii) a sequence with $b' = 0$, $\mu = 0$, and λ taking values in the interval $[\beta \max\{V_b/q, V_k\}, \beta \min\{V_b/\hat{q}, V_k/(1 - \zeta)\}]$.

3. *Derive the associated values of the control variables and of the initial state b.* For each combination $(k, k', \theta, b', \mu, \lambda)$ derived in 1 and 2, compute c, n, b that solve

$$\begin{aligned} U_c(c, k, n) &= \lambda, \\ -U_n(c, k, n) &= \theta\lambda, \end{aligned}$$

and

$$b = qb'^{+} + \hat{q}b'^{-} + g(k', k) + c - \theta n + \tilde{\tau}(\theta).$$

4. *Update* V_b *and* V_k. For each combination $(k, k', \theta, b', \mu, \lambda, c, n, b)$ derived in 1-3, use the envelope conditions

$$\begin{aligned} V_b(b,k,\theta) &= \lambda, \\ V_k(b,k,\theta) &= U_k(c,k,n) - \delta\lambda + \beta \mathrm{V}_k\left(b', k', \theta\right) + \phi_k \mu, \end{aligned}$$

conditions (8)-(9) and the Markov process for θ to compute new values of V_b and V_k. The values of k are on the grid $\left\{k^0, ..., k^m\right\}$ by construction, but the values of b are not in $\left\{b^0, ..., b^n\right\}$, so in this step we use a linear interpolation to compute the values on the grid $\left\{b^0, ..., b^n\right\}$.

Steps 1 to 4 are repeated until convergence of the functions V_b and V_k.

The computation of the optimal policy for the transitional dynamics follow the same approach, except that the functions V_b and V_k have a time index.