Note on "Efficiency of Competitive Search under Asymmetric Information": a Two-Period Example

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October 2006

Consider a two-period example of the dynamic model described in "Efficiency of Competitive Search under Asymmetric Information" with the following simplifying specifications:

1. match-specific disutility shock $\theta \sim U[0,1]$
2. matching function $\mu(\gamma) = \min\{\gamma, 1\}$
3. no discounting: $\beta = 1$
4. zero instantaneous utility to unemployed: $b = 0$

I want to illustrate that the competitive equilibrium is generically constrained efficient, that is it coincides with the social optimal allocation only for a specific initial unemployment rate.

1 Competitive Equilibrium

I first characterize the competitive equilibrium allocation $\hat{\theta}_1^{CE}$, γ_1^{CE}, U_1^{CE}, $\hat{\theta}_2^{CE}$, γ_2^{CE}, U_2^{CE} where $\hat{\theta}_t^{CE}$ is the hiring margin at period t (workers are hired if and only if they

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draw a type $\theta_t \leq \hat{\theta}_t^{CE}$), γ_t^{CE} is the tightness of the market at period t and U_t^{CE} is the continuation utility for unemployed workers at the beginning of period t.

Specializing Proposition 10 in the paper to the finite horizon case, I can solve for the competitive allocation. The important feature of the competitive equilibrium is that at each point in time, agents take as given the future. In particular, in the two-period example, agents at time 1 take as given future equilibrium contracts and market tightness, that is $\hat{\theta}_2^{CE}$ and γ_2^{CE}, which determine U_2^{CE}. Equations (1) and (2) below show that U_2^{CE} affects equilibrium job creation both at time 2 and at time 1. An externality arises because agents do not take into account the impact of U_2^{CE} on job creation at time 1, represented simply by $\hat{\theta}_1^{CE}$ in the example ($\gamma_t^{CE} = 1$ for any t). In the next subsection, I will characterize the social optimum. I will show that the two relations (1) and (2) must be satisfied also by a socially optimal allocation, but the social planner will internalize the impact of U_2^{CE} on $\hat{\theta}_1^{CE}$, generically improving upon the equilibrium.

To characterize the competitive equilibrium allocation, I use Proposition 11 in the paper which shows how γ and $\hat{\theta}$ at each period must maximize the expected utility of unemployed workers subject to a single constraint which comes from the combination of the incentive-compatibility constraints, the participation constraints and the free-entry condition (which is also substituted into the objective function). Going backward:

Period 2. Given that this is the last period of the economy, agents take as given that the continuation utility of being unemployed at the end of the period is going to be 0. Then, the equilibrium $\hat{\theta}_2^{CE}$ and γ_2^{CE} solve

$$U_2^{CE} = \max_{\hat{\theta}_2, \gamma_2} \gamma_2 \left[y\hat{\theta}_2 - \frac{\hat{\theta}_2^2}{2} \right] - \gamma_2 k$$

s.t.

$$\gamma_2 \left[y\hat{\theta}_2 - \hat{\theta}_2^2 \right] \geq \gamma_2 k$$

For any $k > 0$ the constraint is binding. Assume that at the solution $\gamma_2 = 1$ (which will be the case when k is not too big), and the equilibrium $\hat{\theta}_2^{CE}$ solves

$$y\hat{\theta}_2^{CE} - \left(\hat{\theta}_2^{CE}\right)^2 = k.$$

Substituting in the objective, one finds the the following relation between equilibrium job creation and expected utility of being unemployed:

$$\hat{\theta}_2^{CE} = \sqrt{2U_2^{CE}} \tag{1}$$

Trivially, the expected utility of being employed, net from disutility and wages, is zero, that is $V_2^{CE} = 0$.

Period 1. In period 1, agents take as given the continuation utility of being unemployed in the next period, that is U_2^{CE}. Then, the equilibrium $\hat{\theta}_1$ and γ_1 solve

$$U_1^{CE} = \max_{\hat{\theta}_1, \gamma_1} \gamma_1 \left[\left(y - U_2^{CE}\right) \hat{\theta}_1 - \frac{1}{2} \hat{\theta}_1^2 \right] - \gamma_1 k + U_2^{CE}$$

s.t.

$$\gamma_1 \left[\left(y - U_2^{CE}\right) \hat{\theta}_1 - \hat{\theta}_1^2 \right] \geq \gamma_1 k.$$

Assume that at the solution $\gamma_1^{CE} = 1$ (which will be the case when k is not too big), then, as long as $k > 0$, the constraint is binding and $\hat{\theta}_1^{CE}$ satisfies

$$\left(y - U_2^{CE}\right) \hat{\theta}_1^{CE} - \left(\hat{\theta}_1^{CE}\right)^2 = k.$$

Substituting in the objective function I get the relation:

$$\hat{\theta}_1^{CE} = \sqrt{2\left(U_1^{CE} - U_2^{CE}\right)}. \tag{2}$$

Moreover the equilibrium V_1^{CE} and u_2^{CE} for a given u_1 are

$$V_1^{CE} = \beta s U_2^{CE} + \beta (1 - s) V_2^{CE}$$

and

$$u_2^{CE} = u_1 (1 - \hat{\theta}_1^{CE}) + s (1 - u_1).$$

2 Social Planner

I study the Pareto frontier for a given initial rate of unemployment u_1. The Social Planner, for a given u_1, maximizes the *ex-ante* utility of unemployed workers at time 1, U_1, subject to the *ex-ante* utility of employed workers above some given value, that is $V_1 \geq \bar{V}$ and subject to the incentive compatibility constraints, the participation constraints and the resource constraint of the economy. Following section 5.2 in the paper, I analyze the dual version of the problem written in a recursive form. At each period in time, the planner maximize the expected stream of resources subject to two promise keeping constraint respectively for the unemployed and the employed workers, non-negativity constraints for consumption values, the incentive compatibility and the participation constraints. Going backward:

Period 2. Assume that at the solution $\hat{\theta}_2$ is interior, then the planner problem at the beginning of period 2 can be written as

$$\begin{aligned} P_2(V_2, U_2, u_2) \quad = \quad & \max_{\substack{\gamma_2, \hat{\theta}_2, \\ w_2, \underline{w}_2, C_2^V, C_2^U}} \quad u_2 \gamma_2 \left[F(\hat{\theta}_2)(y - w_2) - (1 - F(\hat{\theta}_2)) \underline{w}_2 \right] \\ & -u_2(1-\gamma_2) C_2^U - (1-u_2) C_2^V - u_2 \gamma_2 k \end{aligned}$$

s.t.

$$\begin{aligned} & U_2 = \gamma_2 F(\hat{\theta}_2) w_2 - \gamma_2 \frac{1}{2} \hat{\theta}_2^2 + \gamma_2 (1 - F(\hat{\theta}_2)) \underline{w}_2 + (1 - \gamma_2) C_2^U \\ & V_2 = C_2^V \\ & C_2^U \geq 0; C_2^V \geq 0; \underline{w}_2 \geq 0; w_2 \geq 0 \\ & 0 \leq \gamma_2 \leq 1 \\ & w_2 - \underline{w}_2 = \hat{\theta}_2 \end{aligned}$$

where w_2 is the consumption given to matched workers who are hired at the end of time 2, $\underline{w}_2$ is the consumption given to matched workers who are not hired, C_2^U is the consumption given to unmatched workers and C_2^V is the consumption given to workers who are employed at the beginning of the period. The last equation comes from the incentive compatibility constraints: the worker of type $\hat{\theta}_2$ has to be indifferent between working and getting $w_2 - \hat{\theta}_2$ or reporting any type $\theta_2 > \hat{\theta}_2$ and getting $\underline{w}_2$. Notice that the consumption w_2 has to be constant for all $\theta_2 < \hat{\theta}_2$, otherwise they would all report the type with the highest consumption.

Assume that at the optimum $\gamma_2 = 1$. If U_2 is low enough, that is $U_2 < \frac{1}{2} y^2$ then the non-negativity constraints for $\underline{w}_2$ and C_2^U are binding. I can substitute for w_2 and get

$$\hat{\theta}_2 = \sqrt{2U_2} \tag{3}$$

which is exactly the same relation (1) that holds for the competitive equilibrium allocation (with a general U_2). Moreover,

$$P_2(V_2,U_2,u_2) = u_2\left[y\sqrt{2U_2}-2U_2\right]-(1-u_2)V_2-u_2k.$$

Remark: if we have only one period it is straightforward to check that $\hat{\theta}_2^{CE}$ solves the planner problem for the promised utility U_2^{CE}.

Period 1. Assume that $\hat{\theta}_1$ is an interior solution, then the planner problem at the beginning of period 1 can be written as

$$\begin{aligned}P_1(V_1,U_1,u_1) \quad = \quad & \max_{\substack{U_2,V_2,u_2,\gamma_1,\hat{\theta}_1,\\ w_1,\underline{w}_1,C_1^V,C_1^U}} u_1\gamma\left[F(\hat{\theta}_1)(y-w_1)-(1-F(\hat{\theta}_1))\underline{w}_1\right]\\ & -u_1(1-\gamma)C_1^U-(1-u_1)C_1^V-u_1\gamma_1k+P_2(V_2,U_2,u_2)\end{aligned}$$

s.t.

$$\begin{aligned}&U_1=\gamma_1F(\hat{\theta}_1)(w_1+V_2)-\gamma_1\frac{1}{2}\hat{\theta}_1^2+(1-\gamma_1F(\hat{\theta}_1))(\underline{w}_1+U_2)\\ &V_1=C_1^V+sU_2+(1-s)V_2\\ &u_2=u_1(1-\gamma_1F(\hat{\theta}_1))+s(1-u_1)\\ &C_1^U\geq 0; C_1^V\geq 0; \underline{w}_1\geq 0; w_1\geq 0\\ &0\leq\gamma_1\leq 1\\ &w_1-\underline{w}_1+(V_2-U_2)=\hat{\theta}_1\end{aligned}$$

where w_1, $\underline{w}_1$, C_1^U and C_1^V are the analog of the previous period consumption values. The last equation comes again from the incentive compatibility constraints: the worker of type $\hat{\theta}_1$ has to be indifferent between working and getting $w_1-\hat{\theta}_1+V_2$ or reporting any type $\theta_1>\hat{\theta}_1$ and getting $\underline{w}_1+U_2$, where V_2 and U_2 are the promised utility respectively to employed (net from wages and disutility) and unemployed workers.

Now, suppose that at the solution $\gamma_1=1$ and that the non-negativity constraints for C_1^U and $\underline{w}_1$ are binding. Then the problem can be rewritten as

$$\begin{aligned}P_1(V_1,U_1,u_1) \quad = \quad & \max_{\substack{U_2,V_2,u_2,\hat{\theta}_1,\\ w_1,C_1^V}} u_1\left[\hat{\theta}_1\left(y-\hat{\theta}_1-(U_2-V_2)\right)\right]\\ & -(1-u_1)C_1^V-u_1k+\left\{u_2\left[y\hat{\theta}_2-\hat{\theta}_2^2\right]-(1-u_2)V_2-u_2k\right\}\end{aligned}$$

s.t.

$$\begin{aligned} U_1 &= \frac{1}{2}\hat{\theta}_1^2 + U_2 \\ V_1 &= C_1^V + sU_2 + (1-s)\,V_2 \\ u_2 &= u_1(1-\hat{\theta}_1) + s\,(1-u_1) \\ \hat{\theta}_2 &= \sqrt{2U_2} \end{aligned}$$

Then the first promise-keeping constraint gives me

$$\hat{\theta}_1 = \sqrt{2\,(U_1 - U_2)} \tag{4}$$

which again coincides with equation (2) of the competitive equilibrium (with a general $U_1 - U_2$). The problem becomes

$$\begin{aligned} P_1\,(V_1, U_1, u_1) \;=\; & \max_{U_2, V_2} u_1 \left[\hat{\theta}_1 \left(y - \hat{\theta}_1 - (U_2 - V_2)\right)\right] \\ & - (1-u_1)\,(V_1 - sU_2 - (1-s)\,V_2) - u_1 k \\ & + \left[u_1(1-\hat{\theta}_1) + s\,(1-u_1)\right]\left[y\hat{\theta}_2 - \hat{\theta}_2^2 - k + V_2\right] - V_2 \end{aligned}$$

subject to (3) and (4).

Notice that all the terms containing V_2 cancel out since by the law of motion of u_2

$$V_2 \left[u_1\hat{\theta}_1 + (1-u_1)\,(1-s) - (1-u_2)\right] = 0$$

so the problem can be rewritten as

$$\begin{aligned} P_1\,(V_1, U_1, u_1) \;=\; & \max_{U_2, \hat{\theta}_1, \hat{\theta}_2} u_1 \left[\hat{\theta}_1 \left(y - \hat{\theta}_1 - U_2\right)\right] - (1-u_1)\,(V_1 - sU_2) \\ & -u_1 k + \left[u_1(1-\hat{\theta}_1) + s\,(1-u_1)\right]\left[\hat{\theta}_2 \left(y - \hat{\theta}_2\right) - k\right] \end{aligned} \tag{P}$$

subject to (3) and (4).

The key mechanism that drives inefficiency comes from the fact that the social planner is taking into account the impact of U_2 on both $\hat{\theta}_1$ and $\hat{\theta}_2$, while in the competitive equilibrium the decision of job creation at the second period, $\hat{\theta}_2$, which determines U_2, does not internalize the impact on job creation at time 1, $\hat{\theta}_1$. The problem P above can be written as a maximization problem in one variable, U_2. Then, the first order condition of the social planner problem is

$$
\begin{aligned}
& u_1 \frac{d\hat{\theta}_1}{dU_2}\left(y - 2\hat{\theta}_1 - U_2\right) - u_1\hat{\theta}_1 + (1-u_1)\,s + \\
& + \left[u_1(1-\hat{\theta}_1) + s\,(1-u_1)\right]\frac{d\hat{\theta}_2}{dU_2}\left(y - 2\hat{\theta}_2\right) - u_1\frac{d\hat{\theta}_1}{dU_2}\left[\hat{\theta}_2\left(y-\hat{\theta}_2\right) - k\right] \\
= \; & 0
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
\frac{d\hat{\theta}_1}{dU_2} &= -\left[2\left(U_1 - U_2\right)\right]^{-\frac{1}{2}} = -\hat{\theta}_1^{-1} \\
\frac{d\hat{\theta}_2}{dU_2} &= \left[2U_2\right]^{-\frac{1}{2}} = \hat{\theta}_2^{-1}
\end{aligned}
$$

At the CE, from the constraint binding in the second period it follows that

$$y\hat{\theta}_2 - \hat{\theta}_2^2 = k.$$

Hence, a competitive equilibrium allocation can be a solution for the social planner problem only if it satisfies the first order condition (5) which becomes:

$$-u_1\hat{\theta}_1^{-1}\left(y - \hat{\theta}_1 - U_2\right) + \left[u_1(1-\hat{\theta}_1) + s\,(1-u_1)\right]\hat{\theta}_2^{-1}\left(y - \hat{\theta}_2\right) = 0,$$

that is when the economy starts with the u_1^* that exactly balance the positive effect of U_2 on job creation at time 2 and its negative effect on job creation at time 1. That is, the level u_1^* which solves

$$u_1^*\left(\hat{\theta}_1^{CE}\right)^{-1}\left(y - \hat{\theta}_1^{CE} - U_2^{CE}\right) = \left[u_1^*(1-\hat{\theta}_1^{CE}) + s\,(1-u_1^*)\right]\left(\hat{\theta}_2^{CE}\right)^{-1}\left(y - \hat{\theta}_2^{CE}\right).$$

This is a simple linear equation in u_1^* so I can find the unique value of u_1^* that satisfies this equation (just need to ensure the solution is between 0 and 1). When the economy starts from any level $u_1 \neq u_1^*$, the competitive equilibrium allocation can be improved upon by the social planner, as shown with the next simple numerical example.

3 Numerical example

Parameter values: $y = .9$, $k = .02$, $s = .1$.

Figure 1 plots the function the social planner wants to maximize over the promised utility to unemployed workers, that is

$$\begin{aligned}\Phi(U_2;u_1) &= u_1\left[\sqrt{2\left(U_1^{CE}-U_2\right)}\left(y-\sqrt{2\left(U_1^{CE}-U_2\right)}-U_2\right)\right]-(1-u_1)\left(V_1^{CE}-sU_2\right)\\ &\quad -u_1k+\left[u_1(1-\sqrt{2\left(U_1^{CE}-U_2\right)})+s\left(1-u_1\right)\right]\left[\sqrt{2U_2}\left(y-\sqrt{2U_2}\right)-k\right]\end{aligned}$$

for different values of u_1.

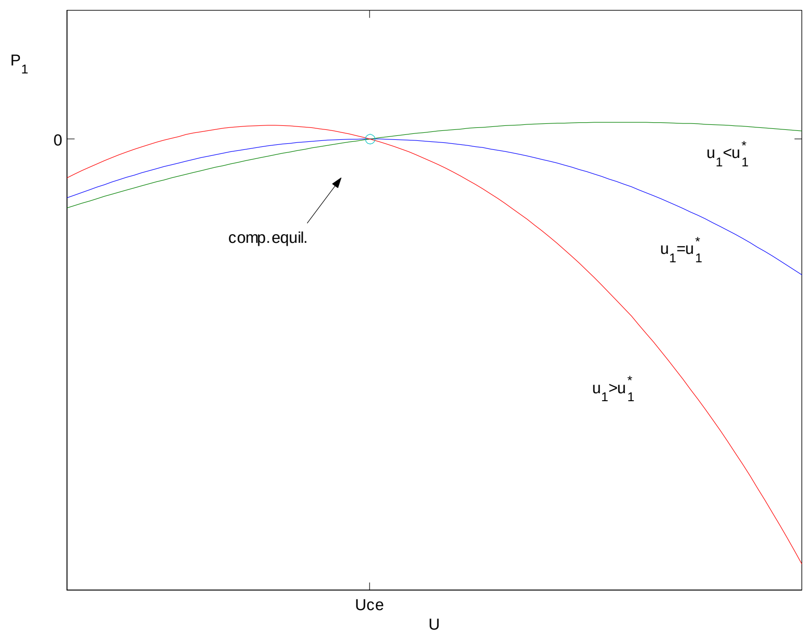

Figure 1.

When $u_1 = u_1^*$ the figure shows that in fact

$$P_1\left(V_1^{CE}, U_1^{CE}, u_1^*\right) = \max_{U_2} \Phi\left(U_2; u_1^*\right) = 0$$

which means that the competitive equilibrium is constrained efficient.
When $u_1 \neq u_1^*$ then

$$P_1\left(V_1^{CE}, U_1^{CE}, u_1\right) > 0$$

and the optimal U_2 is above (below) U_2^{CE} if u_1 is below (above) u_1^*. This shows that the direction of the inefficiency can go in both ways. In particular, increasing U_2 has two opposite effects on welfare: on one hand, it increases the welfare of workers who end up being unemployed at the end of period 1 by increasing $\hat{\theta}_2$, but on the other hand it makes workers who are matched in period 1 worse off by decreasing $\hat{\theta}_1$, because when workers' outside option is higher firms need to pay higher wages and so create less jobs. There exists a specific value u_1^* such that these two effects are exactly balanced. When the initial unemployment rate is higher than that cut-off, then the cost of reducing job creation today dominates and the social planner would like to reduce U_2. When, instead, the initial unemployment rate is lower than that cut-off, then benefits from increasing job creation tomorrow dominates and the social planner would like to increase U_2, as figure 1 shows.