A Theory of Debt Maturity:
The Long and Short of Debt Overhang

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Abstract

Maturing risky short-term debt can impose a stronger debt overhang effect than long-term does, distorting the firm’s investment decisions. We derive the optimal maturity structure based on the trade-off between long-term overhang in good times and (stronger) short-term overhang in bad times. The theory has implications on empirical studies of debt maturity structure, understanding the excessive defaults and underinvestment during recessions, market-based pricing of credit lines, and firm’s cash holdings.

Key words: Wealth transfer, short-term debt crisis, underinvestment, endogenous default.
1. Introduction

Myers (1977) shows that risky debt that matures in the future leads to underinvestment today. The insight is that part of the cash flows generated by investment goes to debt holders at maturity; unfortunately the equity holders who make the investment decision will not internalize this benefit. The truncation of cash flows (and implied sharing of them) can distort investment incentives. Myers (1977) therefore suggests the solution of short-term debt to the debt overhang problem, because if all debt matures before the investment opportunity, the firm can make the investment decision as if an all-equity firm.

The short-term debt in Myers (1977) is better viewed as debt that has matured (safely) yesterday before the investment decision, rather than the one that is going to mature soon. We stress the importance of timing here, as this paper studies the situation where the firm is making a serious investment decisions (say, expansion, maintenance, or even default) given both short-term and long-term debt maturing in the future. To our best knowledge, this empirically relevant situation has not been explored in the literature.

This paper focuses on the impact of short-term debt overhang on the firm’s long-term investment decisions. Our four-date model features two new elements relative to Myers (1977). First, the firm’s debt maturity structure choice (debt maturity at \( t=1 \) or \( t=2 \)) is made at \( t=-1 \) before the information of asset-in-place gets realized at \( t=0 \). Second, the short-term debt is going to mature but still before the firm’s investment decision at \( t=0 \). Specifically, at \( t=0 \) the firm is considering taking a long-term investment opportunity, i.e., its cash flows will realize at date 2. However, the firm has not only the long-term debt which matures at date 2, but also the short-term debt maturing at date 1 which requires refinancing. Since new investment involves wealth transfers to both debts, both long-term and short-term debts may impose overhang effect on the investment at date 0. Interestingly, we find that the relative strength of overhang effects caused by the long-term and short-term debt is state-contingent, where the state is the firm’s asset-in-place (or can be equivalently interpreted as the firm’s value from future cash flows).

In general, from Myers (1977) we know that short-term debt imposes overhang when it becomes risky (say the firm’s asset-in-place deteriorates). The goal of this paper is to establish the trade-off between short-term and long-term overhang at the time of maturity decision at \( t=-1 \). Equivalently, we show that short-term overhang effect dominates the long-term overhang when short-term debt turns risky due to interim bad news on firm’s asset-in-place.
Given the above timing structure that the firm issued debt before certain significant news arrives regarding the value of asset-in-place, we identify the *increasing leverage* effect due to short-term debt. Specifically, when the interim bad news hits, the short-term debt which does not share risk with equity holders will have a greater debt value (higher leverage) compared to the long-term debt. This in turn causes a greater overhang due to short-term debt in bad times.

In examples studied in Section 2, we consider the firm takes either exclusively short-term or long-term debt. When the asset-in-place value is high, the short-term debt can easily get refinanced. As a result, riskless short-term debt imposes no overhang. In contrast, the long-term debt is risky due to greater risk in the long-run, leading to a positive overhang. However, when the asset-in-place value deteriorates, things might change. Compared to long-term debt, the value of short-term debt drops less, leading to a higher leverage and greater overhang. In our example, given bad news the firm always has trouble in refinancing its short-term debt at interim date \( t=1 \) if it does not invest. In fact, the risky short-term debt imposes 100% leverage on the firm, leading to the greatest possible overhang. In contrast, as the firm might still survive at \( t=2 \) when sufficiently good news hit, the long-term debt has a lower leverage and therefore less overhang.

The economic mechanism underlying the stronger short-term overhang is simple. The core of debt overhang is caused by wealth transfer from equity holders to debt holders. Because short-term debt is going to be paid earlier than the long-term debt, there is less uncertainty resolved over the shorter time, and the wealth transfer to potentially risky short-term debt holders (through new investment) will be greater. In our example with low asset-in-place value, the firm with short-term debt is forced to default at date 1 even if the firm invests at \( t=0 \). This immediately implies that short-term debt holders take the firm over and claim all the investment benefits from that point on. In contrast, if the firm has used long-term debt instead, then there is no default at date 1, and equity holders can recoup some investment benefit at date 2 if the firm’s asset-in-place improves.

In Section 2.3 we also give an example where risky short-term debt (in bad times) impose greater overhang even if we control for leverage, i.e., keep the same market value for both long-term and short-term debt. This implies that even if there is no significant news about asset-in-place that arrives between the debt maturity decision and the investment decision, it is still possible that the firm would prefer to issue long-term debt to reduce overhang, especially when the firm’s asset-in-place exhibits higher volatility for low asset value levels.
Our finding implies that short-term debt is not a free lunch in coping with debt overhang. Although riskless short-term debt (when the firm’s asset-in-place value is high) impose no overhang, risky short-term debt (when the firm’s asset-in-place deteriorates) does more evil than long-term debt in dwarfing the firm’s investment incentives. Therefore, the firm faces a trade-off in choosing the optimal debt maturity structure before the state containing about asset-in-place realizes. Section 3 presents a formal model which endogenizes the debt maturity structure (chosen at t=−1) based on the trade-off just illustrated.

In our model, the interpretation of “investment” can be spending capital to either establish new projects, or keep old projects alive. For instance, “maintenance” is a form of investment that requires capital expenditure to keep the existing project operating efficiently, and “foregoing maintenance” represents underinvestment. The extreme version of “foregoing maintenance” is just “default,” where the firm essentially gives up the old project. In fact, the endogenous default, which usually occurs given a sufficiently low asset-in-place value, is a symptom of short-term debt overhang (see Leland (1994) and related literature on endogenous default).2 This is because the new financiers understand that to prevent the firm from defaulting, the fresh capital that they put in pays the maturing short-term debt first, resulting in a full subsidy to short-term debt holders. Although it is true that bailing out the firm also benefits long-term debt holders who have claim on the firm’s future cash flows, the subsidy to long-term debt holders will not be as large, as new financiers can recoup some benefit if the firm recovers tomorrow.

Debt overhang has been an active research topic since Myers (1977).3 This paper emphasizes the role of debt maturity on debt overhang. We show that the short-term overhang is in a zero-one nature. To be precise, when the firm’s asset-in-place value is high and short-term debt is riskless, short-term debt imposes no overhang effect. However, once the firm has experienced a series of negative shocks and short-term debt becomes risky, the short-term debt overhang quickly becomes overwhelming. Therefore, relying on short-term debt can cause higher votalility in overhang (i.e., zero overhang in good times, while greater overhang in bad times).

In many financial crises, short-term debt is often implicated as contributing factor to defaults. However, often times it is attributed to a “run” by short-term debt (Diamond and Dybvig (1983), Diamond and Rajan (2001), Goldstein and Pauzner (2005), and He and Xiong (2009)).

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2 The interpretation of endogenous default given debt burden as “underinvestment” due to debt-overhang, is mentioned in, for example, Lambrecht and Myers (2008) and He (2010).

3 To name a few recent studies on debt overhang, see Philippon and Schnabl (2009) and Diamond and Rajan (2010).
Meanwhile, researchers acknowledge the debt-overhang effect in bailing out financial firms, but mostly attribute the overhang effect to existing long-term debt. By showing that the distortion due to overhang is not a feature of long-term debt exclusively, this paper suggests that the short-term overhang can be another important contributing factor to the current crisis. In fact, the results in Veronesi and Zingales (2009) suggest that the wealth transfer in the recent government bailout of financial firms concentrates on the short-term debt.4

Moreover, the prediction that short-term debt overhang leads to underinvestment is consistent with the empirical findings in Duchin, Ozbus, and Sensoy (2009) and Almeida et. al (2009) who study firms’ underinvestment during the 2007/2008 crisis. Almeida et. al (2009) design a quasi-natural experiment, in which they examine two groups of firms with similar amount of total long-term debt, but with different current portion of what was originally long-term debt when it was issued. In this sense, they are comparing two otherwise identical firms but with different maturity structure. They find that firms with larger current portion of long-term debt cut back investment more than those with smaller current portion, and attribute this result to the disruption of credit market during the 2007/2008 crisis. However, in our model even though the ability to raise new funds from financial markets is perfect, the larger current portion of long-term creates stronger short-term overhang effect, which can also leads firms to cut back their investment. Therefore, their empirical design is not perfect in separating the story of disruption of credit market from that of short-term debt overhang (which is driven by lower firm fundamental).

Our theoretical results have important implications for empirical testing of the Myers (1977) debt-overhang theory, especially the prediction that growth firms (presumably with more investment opportunities) should use more short-term debt. The existing empirical evidence on this prediction has been mixed. For instance, Barclay and Smith (1995) and Guedes and Opler (1996) find a negative relation between maturity and growth opportunities, a result consistent with Myers (1977). However, these studies do not control for firm leverage. Since leverage usually is positively correlated with maturity, their finding could be due to that growth firms tend to have lower leverage. In contrast, Johnson (2003) argues that maturity and leverage are jointly endogenously determined. Using the standard two-stage instrumental variables regression

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4Veronesi and Zingales (2009) evaluate US Treasury secretary Paulson’s plan announced after the Lehman bankruptcy in October 2008. Based on the banks’ CDS prices for debts with different maturities, that paper constructs the Bank Run index, and finds that the index drops dramatically after the announcement of Paulson’s plan. Because the Bank Run index is negatively related to the CDS price of short-term debt, this empirical finding offers supporting evidence to our theory.
technique, Johnson finds a positive relation between maturity and growth opportunities. In light of our theory, these mixed results are not surprising, because 1) early default for growth firms might be more costly which pushes optimal maturity structure toward long-term, and 2) the presence of short-term and long-term debt has state-contingent overhang effect on investment.

The optimal maturity structure in our paper is based on the trade-off due to long-term and short-term overhang. It is different from existing theories of optimal maturity structure which focus on the disciplinary role to short-term debt in curbing the managers’ asset substitution or other misbehavior, e.g., Calomiris and Kahn (1991), Diamond and Rajan (2001), Flannery (1994) and Leland (1998), or private information of borrowers about their future credit ratings, e.g., Flannery (1986) and Diamond (1991). He and Xiong (2010) study the impact of bond market illiquidity on credit risk; the trade-off in that paper is based on the stronger overhang effect in rolling over short-term debt and its greater liquidity in the secondary market.

The rest of paper is organized as follows. We give an example in Section 2 which illustrates the key intuition of this paper. Section 3 gives the formal model, and Section 4 provides discussion and extensions. In Section 5 we conclude.

2. Examples and Brief Discussion

2.1 Example Structure

The following examples deliver the main intuition of this paper. We are standing at t=0, where the firm has some asset-in-place, and is considering making a new investment which requires new outside financing (as the firm has no cash holdings; see discussion in Section 4.4).

At t=0, equity holders face a debt maturity structure which was determined at t=-1, a decision that will be formally studied in Section 3. The premise here is that we will focus on the case that there is some significant amount of information reveals after the maturity decision. Although

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5 In fact, Stohs and Mauer (1996) control for firm leverage directly (i.e., unfitted value) and also find a significantly positive relationship between maturity and growth opportunities.
6 Hart and Moore (1994, 1998) study the optimal debt maturity that persuades the entrepreneur to pay out cash flows; their analysis is based on renegotiation, which is ruled out in this paper for debt overhang to exist in the first place. Brunnermeier and Yogo (2009) stress the option value of using short-term financing so that the firm can readjust the debt maturity before the firm has experienced sufficiently negative shocks. In our model, the short-term debt matures after the sufficiently negative shock arrives. And, Brunnermeier and Oehmke (2009) study the maturity rat race between creditors; there, short-term creditors impose negative externalities on long-term creditors, leading to excessive short-term debt in equilibrium. Benmelech (2006) argues that entrenchment can lead the self-interested manager to take long-term debt, and provide supporting evidence for this theory.
Case 1: Riskless ST debt

Case 2: Risky ST debt

Figure 1: A binomial example with two cases. Following the previous node, each branch occurs with a probability of 0.5. Case 1 (2) has a high (low) asset-in-place value so the short-term debt will be riskless (risky).

Section 2.3 also investigates the possibility of maturity choice at t=0, we believe the information setting of maturity choice at t=-1 is more empirically relevant.

The agents are risk neutral and the interest rate is zero. Short-term debt will be maturing at t=1, while long-term debt will be maturing at t=2. Importantly, since there are no interim cash flows, short-term debt needs refinancing at t=1. To study debt overhang, we assume existing debt to be senior to any new financings. We rule out renegotiation, and bankruptcy (if occurs) cost is zero.

To clearly see the trade-off, let us consider the case where the firm is allowed to take either exclusively short-term or long-term debt. The full analysis in Section 3 will consider the optimal mix of long-term debt and short-term debt.

The firm’s asset-in-place follows a standard binomial tree depicted in Figure 1. With probability 0.5 the state u (d) occurs at t=1, and with probability 0.5 further uncertainty resolves. In state dd the asset-in-place has a value of 1. In state ud or du, the asset-in-place’s value is $X$, and in uu state the value is $X^2$. The left (right) panel has a higher (lower) asset-in-place $X$, which determines whether the short-term debt is risky or not.

Suppose that the new investment has a cost of $I < 1$ at t=0 leads to a constant payoff of 1 at t=2, which implies a positive NPV of $1- I$. When equity holders invest at t=0, debt overhang implies that part of the investment payoffs goes to the debt holders (i.e., wealth transfer).
Equivalently, this wealth transfer is reflected in the value increment of debt (either short-term or long-term) due to the new investment. Since there is no other dead-weight loss in this example, equity holders will invest if and only if the NPV of this investment, i.e., $1 - I$, exceeds the wealth transfer to the debt holders.

2.2 Example 1: Increasing Leverage Given Bad Interim News

Throughout this example, the short-term debt has a face value of $F_1=11$, and the long-term debt has a face value of $F_2=14.75$. These face values allow the firm to raise the same amount of debt financing independent of debt maturity at $t=-1$, if we assume that Case 1 (good state) and Case 2 (bad state) occurs equally likely standing at $t=-1$. Also note that since $F_2$ is above the asset-in-place at state $dd$ (which is 1) for both cases, long-term debt is always risky.

2.2.1 Analysis

Any debt, if it is riskless, imposes no overhang. Therefore, the existence of short-term overhang depends on the value of asset-in-place. In the first (second) case, the asset-in-place has a relatively high (low) value and the short-term debt is riskless (risky). Throughout this paper we say the debt is risky if the firm’s asset-in-place is insufficient to pay debt holders in full, given no investment opportunities. We adopt this convention because we are interested in the firm’s investment incentives once it faces investment opportunities.

Case 1: Riskless short-term debt in the left panel of Figure 1

Short-term debt only. The short-term debt with face value $F_1=11$ can be refinanced even at the state $d$, as the asset-in-place there has a market value of $0.5*(21+1)=11$. Hence the short-term debt is riskless and has a market value of 11 at $t=0$. Because the value of riskless short-term debt cannot be improved further, the wealth transfer to debt holders is zero by taking the investment.

Long-term debt only. Long-term debt with face value $F_2=14.75$ is risky, which leads to debt overhang. Without investment, the long-term debt value is $0.75*14.75+0.25=11.3125$. But with investment, the long-term debt value increases to $0.75*14.75+0.25*2=11.5625$. As a result, the wealth transfer to long-term debt involved in the new investment is 0.25, which implies that equity holders will pass on investment opportunities with NPV less than 0.25. This example illustrates the long-term overhang effect discussed in Myers (1977).

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7 This property of equal debt financing holds either with investment or without investment at $t=0$. Since in this example we focus on the impact of the existing leverage and debt maturity on firm’s investment incentives, we ignore the feedback from investment decisions to debt valuation. In Section 3 we fully consider the feedback effect.
### Table 1: Leverage Effect of Short-term Debt Overhang

<table>
<thead>
<tr>
<th></th>
<th>Case 1: High Asset-in-Place, Riskless ST Debt</th>
<th>Case 2: Low Asset-in-Place, Risky ST Debt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ST Debt only</td>
<td>LT Debt only</td>
</tr>
<tr>
<td>Asset-in-place Value at t=0</td>
<td>121</td>
<td>121</td>
</tr>
<tr>
<td>Debt Face Value</td>
<td>11</td>
<td>14.75</td>
</tr>
<tr>
<td>Debt Value at t=0 (w/o investment)</td>
<td>11</td>
<td>11.3125</td>
</tr>
<tr>
<td>Leverage at t=0 (w/o investment)</td>
<td>9%</td>
<td>9.3%</td>
</tr>
</tbody>
</table>

#### Panel A: Leverage and Credit Spread w/o Investment

#### Panel B: Debt Overhang and Wealth Transfer

<table>
<thead>
<tr>
<th></th>
<th>ST Debt only</th>
<th>LT Debt only</th>
<th>ST Debt only</th>
<th>LT Debt only</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt Value at t=0 with investment</td>
<td>11</td>
<td>11.5625</td>
<td>7.25</td>
<td>6.6875</td>
</tr>
<tr>
<td>Wealth Transfer to Debt holders</td>
<td>0</td>
<td>0.25</td>
<td>1</td>
<td>0.75</td>
</tr>
<tr>
<td>Investment Policy</td>
<td>NPV$\geq0$</td>
<td>NPV$\geq0.25$</td>
<td>NPV$\geq1$</td>
<td>NPV$\geq0.75$</td>
</tr>
</tbody>
</table>

**Case 2: Risky short-term debt in the right panel of Figure 1**

**Short-term debt only.** Now in this case the asset-in-place is low. Without investment, the short-term debt with face value 11 cannot be refinanced in either states $d$ or $u$—even at state $u$ the asset-in-place only has a value of $0.5*16+0.5*4=10<F_1=11$. Therefore the firm will default, and the date-0 market value of short-term debt is just the firm’s asset-in-place value 6.25.

How much does the short-term debt gain if the firm invests? Since the firm is totally underwater in this case (100% leverage as shown in Table 1), one would expect short-term debt to gain all of the initial increment of value from new investment. Here, if the firm invests at t=0, at state $d$ the firm defaults (and short-term debt holders receive $0.5*5+0.5*2=3.5$), while at $u$ (with an asset-in-place value 11) the short-term debt can just get refinanced. As a result, the short-term debt value at t=0 increases from 6.25 to 7.25. The wealth transfer from equity holders to short-term debt holders is the entire investment project payoff 1, and equity holders will never invest.
Long-term debt only. The analysis for long-term debt is similar to the Case 1. Since the wealth transfer to long-term debt holders due to investment is 0.75, equity holders will pass on investment opportunities with NPV less than 0.75.

The wealth transfer to short-term debt (1) is greater than that to long-term debt (0.75). Although long-term debt benefits only at the bottom states \{du, ud, dd\} with probability 0.75, short-term debt benefits on all t=2 states. In general, there are two mechanisms that short-term debt holders receive wealth transfer. The first mechanism is default, which is at work in state d at t=1 in Case 2. There, since pledging out entire future asset payoffs is insufficient to pay down the short-term debt, the short-term debt holders take over the defaulting firm to claim all date-2 cash flows from then on. The second mechanism is refinancing, i.e., the firm raises new financings (say, new equity) to pay the date 1 short-term debt in full, and these new financiers will recover these payments from asset payoffs at date 2. In our example, refinancing works at state u, which allows the short-term debt holders to obtain the additional cash flows at state uu. As clear from the example, it is the refinancing mechanism that allows short-term debt to claim the entire cash flows at state uu, and in turn impose stronger overhang than long-term debt.

2.2.2 Increasing leverage effect and a log-normal example

Recall that in this example, fixing the investment policy (with or without investment) the debt value at date -1 is the same across both maturity structures. However, when the news about the asset-in-place arrives at t=0, because short-term debt tends to be a “hard” claim which do not share gain/loss with equity holders, the value of short-term debt is less sensitive to the new information than the long-term debt (similar to Myers and Majluf (1984)). In our example, after good (bad) news at t=0, the short-term debt value rises (drops) by 2.375, while the long-term debt value rises (drops) by 2.6875. These different sensitivities to news translate to different contingent amounts of leverage for the different maturities; particularly, contingent on bad news, short-term debt with higher value exhibits a higher leverage and in turn stronger overhang. In fact, in Case 2, the asset-in-place deteriorates so much that short-term debt produces 100% leverage, which generates strongest overhang. As a result, short-term debt which is raised before the information may hurt the firm’s ex post investment incentives more than the long-term debt does. Apparently, this will affect the optimal maturity choice at t=-1.

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8 For example, without investment, the date -1 debt value is \((11+6.25)/2=(11.3125+5.9375)/2=8.625\) as both cases are equally likely.
This “increasing leverage” effect, which only relies on the fact that short-term debt is less informationally sensitive than long-term, is quite general. For further illustration, consider another example based on the standard Black-Scholes framework. Suppose that the firm’s asset-in-place value $V_2$ at $t=2$ is

$$V_0 \exp \left( \frac{-\sigma^2}{2} + \tilde{X}_2 - \frac{\sigma^2}{2} \right)$$

where $\tilde{X}_1$ and $\tilde{X}_2$ follow i.i.d. normal distributions with standard deviation of $\sigma$. This implies that the firm value at $t=0$ is $V_0$, and at $t=1$ is $V_0 \exp \left( \frac{-\sigma^2}{2} \right)$.

As before, let us consider short-term debt with face value $F_1$ or long-term debt with face value $F_2$. Since debt value is firm value minus equity value (which is a call option $E(V_0)$ on the firm value), the debt overhang, as wealth transfer to debt holders, can be measured as the marginal improvement of debt value when the firm considers investment to improve $t=0$ firm value $V_0$:

$$\text{Overhang}(V_0, F_1, t) = 1 - \frac{\partial E(V_0, F_1, t)}{\partial V_0} = 1 - \Delta(V_0, F_1, t).$$

where $\Delta$ is the delta-hedge for the call option in the Black-Scholes formula. Economically, imagine the firm’s incremental investment projects to be that, each dollar of expansion on $V_0$ only requires $1 - b \in (0, 1)$ dollar of investment, i.e., the project’s NPV is $b > 0$. Now because equity holders understand that each unit of expansion only benefits them by $\Delta$, they invest if their investment outlay exceeds the investment benefit, i.e., $1 - b \geq \Delta$. In other words, equity holders pass investment projects with NPV $b \leq 1 - \Delta$. Hence, the above wealth transfer measure $1 - \Delta$ directly gives the NPV threshold of the investment policy taken by equity holders. Finally, as any investment requires some positive outlay, an overhang of 1 would imply to pass on all investments (i.e., never invest).

In Figure 2, to examine the increasing leverage effect, we vary firm value $V_0$ while fixing the face values $F_1$ or $F_2$. We set $(F_1, F_2)$ to give the same value for short-term and long-term debts when $V_{-1} = 1$. A deviation of $V_0$ from $V_{-1} = 1$ can be viewed as new information arrives about the firm’s asset-in-place at $t=0$. The left panel in Figure 3 illustrates that when bad news hits ($V_0$ drops below 0.7), short-term debt imposes stronger overhang than long-term debt. This is the
increasing leverage effect: As illustrated at the right panel, short-term debt value $D_1$ is less sensitive to the bad news (i.e., flatter than the long-term debt $D_2$), and $D_1 > D_2$ when $V_0$ is low.

The general message in Figure 2 is that short-term debt features a less volatile debt value, which translates to more volatile firm’s leverage. In turn, a more volatile leverage leads to more volatile overhang---the corresponding equity holders’ NPV threshold $1-\Delta$ is more volatile. In particular, when bad news hits, the higher leverage leads to a greater short-term debt overhang, and equity holders are more likely to pass investment projects with positive NPV. To the extreme, consider the ultra-short debt with time-to-maturity $t$ converging to 0. Once $V_F < 0$, $(V_F, F, t)$ goes to 0, implying an overhang of $1- \Delta = 1$. This corresponds to the situation where equity holders get zero and debt holders who have immediate claims on the insolvent firm receives everything, which is the greatest debt overhang!

2.3 Example 2: Overhang differences by maturity for given leverage

2.3.1 Analysis

We have established one dark side of short-term debt, in that the (risky) short-term debt impose a stronger overhang when some bad news arrives after raising the debt and the firm becomes more levered. This is the information structure studied in the main model in Section 3. As in reality firms who face existing debts are experiencing sequence of productivity shocks and carry out investment along the way, this framework is relevant in studying the short-term overhang.
One can modify the information structure and ask a further question: what if the firm chooses the maturity structure right after the bad news hits at $t=0$, as opposed to at $t=-1$ before news arrives? Put differently, once we force the short-term and long-term debts to have the same value (therefore eliminating the increasing leverage effect studied in Section 2.2), can short-term debt still impose stronger overhang?

There are three reasons why this question is interesting. First, it clarifies that a driving force of stronger short-term overhang that can be independent of leverage. Second, even the formal model in Section 3 later studies a different information structure where maturity choice is made before any news shock arrives (at $t=-1$), this effect will be present there. It is because if the firm wants to substitute short-term debt with long-term debt (maintaining the same total debt value) even given bad news at $t=0$, then the firm definitely would like to do so at $t=-1$. Last but not the least, the answer to this question offers the guidance for firms how to rebalance their maturity structure in bad times (when all debt issued might need to be risky).

### Table 2: Stronger Short-term Debt Overhang For a Fixed Leverage

<table>
<thead>
<tr>
<th></th>
<th>Case 2, Risky ST Debt</th>
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<tbody>
<tr>
<td></td>
<td>ST Debt only</td>
<td>LT Debt only</td>
</tr>
<tr>
<td>Panel A: Leverage and Credit Spread w/o Investment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Asset-in-place Value at $t=0$</td>
<td>6.25</td>
<td>6.25</td>
</tr>
<tr>
<td>Debt Face Value</td>
<td>3.5</td>
<td>11/3</td>
</tr>
<tr>
<td>Debt Value at $t=0$ (w/o investment)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Leverage at $t=0$ (w/o investment)</td>
<td>48%</td>
<td>48%</td>
</tr>
<tr>
<td>Panel B: Debt Overhang and Wealth Transfer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Debt Value at $t=0$ with investment</td>
<td>3.5</td>
<td>3.25</td>
</tr>
<tr>
<td>Wealth Transfer to Debt holders</td>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>Investment Policy</td>
<td>NPV $\geq 0.5$</td>
<td>NPV $\geq 0.25$</td>
</tr>
</tbody>
</table>
The following example given in Table 2 provides a positive answer. In words, it is possible that given that short-term debt and long-term debt have the same debt value at t=0, the short-term debt impose stronger overhang. Consider the Case 2 with asset-in-place depicted in Figure 1, but with F_1=3.5 or F_2=11/3<4. One can easily check that both debts have a value of 3. However, short-term overhang is 0.5, which is greater than long-term overhang 0.25.

Unlike the first effect identified in Section 2.2, this stronger effect is present only for some probability distributions of firm value. The driving force of this effect is as follows. The firm with interim bad news at t=1 (i.e., at state d) cannot survive under short-term debt, and all investment benefits following on this path (du and dd) go to the short-term debt holders. However, under long-term debt, the firm has some chance to come back at t=2 (the state du), which allows equity holders to recoup the investment benefit at state du.

As suggested by this example, the reason that the (risky) short-term debt could impose stronger overhang, even controlling debt value, is because there is less uncertainty to be resolved over a shorter time horizon. Through refinancing or default, short-term debt holders may capture the investment benefit in the subsequent states---even for the states (i.e., du state) after sufficiently good follow-up news that long-term debt holders are not able to capture.

Of course, the opposite logic might work at state u---i.e., things might go worse after u---and this leads to stronger overhang for long-term debt. In our example, this effect is absent as the negative shock following date 1 good news is not that severe (i.e., at ud state the asset value is 4>F_2). However, this observation does suggest that, in general, short-term debts impose stronger overhang when the firm may improve dramatically after interim bad news, while there will be insignificant shocks after interim good news. Based on this idea, in the next subsection we give an example that this effect holds in a modified Black-Scholes case.

### 2.3.2 An Example with State-Dependent Volatility

To illustrate how the distribution of value affects the severity of overhang for the two maturities, consider the following modification of Black-Scholes framework as a continuation of Section 2.2.2. The firm’s asset-in-place value at t=2 is

\[
V_0 \exp \left( \bar{X}_1 - \frac{\sigma_1^2}{2} + \bar{X}_2 - \frac{\sigma_2^2}{2} \right)
\]

---

9 We sincerely thank Stewart Myers for pushing us on this direction.
where \( \bar{X}_1 \) and \( \bar{X}_2 \) follow normal distribution with variances \( \sigma_1^2 \) and \( \tilde{\sigma}_2^2 \), respectively. Recall that the value on date \( t=1 \) is \( V_0 \exp\left(\bar{X}_1 - \frac{\sigma_1^2}{2}\right) \). We allow the volatility \( \tilde{\sigma}_2^2 \), to be dependent on the \( t=1 \) realization \( \bar{X}_1 \). Particularly, we set

\[
\tilde{\sigma}_2 = \begin{cases} 
\sigma_L & \text{when } \bar{X}_1 > Q \\
\sigma_H & \text{when } \bar{X}_1 \leq Q
\end{cases}
\]

where \( \sigma_L \leq \sigma_H \). This corresponds to the situation where volatility is higher in low fundamental states, which is a common feature in many macro models with financial frictions. In fact, this pattern can be generated by the existence of volatility that are not scaled with the asset value (say, randomness in the fixed cost.)

We aim to find cases that even when \( D(V_0, F_1, 1) = D(V_0, F_2, 2) \), the short-term debt imposes stronger overhang than long-term debt so that \( 1 - \Delta(V_0, F_1, 1) > 1 - \Delta(V_0, F_2, 2) \). The following proposition formally states this result holds when 1) \( \sigma_L = 0 \) while raising \( \sigma_H \) to be above zero; and 2) \( F_2 = V_0 \exp\left(\frac{Q - \sigma_L^2}{2}\right) \) so that the conditional volatility becomes higher exactly when long-term debt is at the money at \( t=1 \).
Proposition 1. Suppose that \( F_2 = V_0 \exp \left( Q - \frac{\sigma_1^2}{2} \right) \) and \( \sigma_L = 0 \). Then when \( \sigma_H \) increase above zero, \( 1 - \Delta(V_0, F_1, 1) > 1 - \Delta(V_0, F_2, 2) \) for small \( \sigma_H \).

See the proof in Appendix A.1.1. Further in Appendix A.1.2 we give details in reducing the two-dimensional integrals involved in calculating long-term debt value \( D(V_0, F_2, 2) \) and its overhang to some one-dimensional ones. This allows us to use Matlab built-in function \textit{quadgk.m} to numerically evaluate them, with accuracy level \( 10^{-12} \).

Figure 3 plots a numerical example when we increase \( \sigma_H \) from \( \sigma_H = 0.01 = \sigma_L \) to \( \sigma_H = 0.12 \). The left panel plots the debt face values where we adjust \( F_2 \) to ensure the same market value for both debts. The right panel plots the resulting short-term and long-term overhang. Since we fix \( \sigma_1^2 \) and \( F_1 \), the short-term debt overhang is a constant. However, when we increase the volatility \( \sigma_H \) in low fundamental state \( \tilde{X}_1 < Q \), the long-term debt overhang drops below the short-term debt overhang. The reason is simple: given the greater volatility after bad news, it is more likely for the firm’s value to jump back above the long-term debt face value \( F_2 \), and the probability mass that \( V_2 \) below \( F_2 \) (therefore overhang) becomes smaller.

2.3.3 Discussion: Maturity Adjustment

The above findings have important implications for firms who are deciding maturities along with investment at \( t=0 \), or considering readjusting its maturity structure at \( t=0 \), conditional on the firm’s value of asset-in-place. Clearly, when the asset-in-place is relatively high, the firm would be better off by replacing risky long-term debts with riskless short-term debts, as any riskless debt imposes no overhang. But once the firm’s asset-in-place deteriorates, risky short-term debt can impose stronger overhang than the long-term debt does, if the firm’s assets displays higher volatility following bad shocks. /we need to give some industry examples./ At this scenario, the firm would be better off by replacing long-term debts with risky short-term debts with the same market value.\(^{10}\)

\(^{10}\) It is important to note that the maturity adjustment studied here implicitly involves renegotiation with debt holders, which is ruled out in our analysis from the beginning (to focus on debt overhang). To see this, “controlling debt values” means that the firm pays the debt holders their market value, not face value, for giving up the debt claim. Or, the debt must be non-standard in which some covenants give the firm an option to make this adjustment.
3. The Model

We have illustrated the state-contingent effect of short-term overhang, i.e., that short-term debt imposes no overhang when the firm’s asset-in-place value is high while stronger overhang when the value is low. This suggests a trade-off when the firm makes the debt maturity decision at t=-1, and now we consider a formal model which incorporates the feedback effect from investment policy to t=-1 debt value. At t=-1 the firm needs to raise debt financing $D_{\text{target}}$ for initial investment. To focus on debt maturity choice, we fix the debt level $D_{\text{target}}$ in the main analysis. Section 4.3 endogenizes $D_{\text{target}}$ by considering an entrepreneur who requires outside capital to finance this project.

The risk-free rate is zero, and agents are all risk neutral. The credit market is competitive so that all debt is fairly priced. In this model shocks are all independent. All information is public. No cash flows from the asset occur before date 2, and we ignore cash holding.\footnote{We study the implication of cash holdings in this model in Section 4.4.}

After the firm chooses maturity structure at t=-1, at t=0 it faces the situation described in Section 2, with one simplifying modification in the asset-in-place tree. As shown in Figure 4, with probability 0.5 the state $u$ occurs at t=1, and there is no future uncertainty. With probability 0.5 the state $d$ occurs at t=1; following that, at t=2 the state might be $du$ ($uu$) or $dd$ ($ud$) with equal probabilities.

There are two possible states of nature at t=0, $S=G$ (good) or $S=B$ (bad). As considered in Section 2, the firm’s asset-in-place is higher (lower) in good (bad) state. Specifically, asset-in-place generates cash flows $2X^S$ in state $Suu$, $X^B$ in state $Sdu$ or $Sud$, and 0 in state $Sdd$.

The firm faces investment opportunity $I^S$ at t=0, which yields a constant payoff $Y^S$ at t=2, $S \in \{G,B\}$. This implies that the investment has a positive NPV of $\gamma^S = Y^S - I^S$. Throughout we will assume that the NPV $\gamma^S$ is relatively moderate so that the necessary condition for investment at t=0 state $S$ is successful refinancing at state $Su$.

3.1 Short-term and Long-term Debt Contracts

We only consider standard debt contracts. The short-term (long-term) debt, with t=-1 market value $D_1$ ($D_2$) has a face value $F_1$ ($F_2$). We can summarize the firm’s debt policy as $(F_1,F_2)$ so
that $D_1 + D_2 = D_{\text{target}}$. We impose the following assumption throughout the paper to focus on the
debt overhang problem.

**Assumption.** Neither short-term debt nor long-term can be renegotiated (perhaps because debt
holders are diverse). The long-term debt has a covenant stating that any new financing will be
junior to the long-term debt.

The firm may default if it cannot repay its maturing debt. We impose no bankruptcy cost in
the main model. Perhaps more importantly, as we are only interested in the wealth transfer to
total debt, our analysis does not depend on the seniority rule between the long-term and short-
term debts in bankruptcy.

### 3.2 Solving the Model

#### 3.2.1 Good State G

Consider the state $Gd$ at $t=1$ first. Without new investment, the maximum new financing that
the firm can obtain, i.e., refinancing capacity, is $\frac{1}{2}\max(X^G - F_2, 0)$. The new financing can be

---

Figure 4: Timeline of the model.
raised from existing equity holders/managers or outside equity holders.\textsuperscript{12} Since it is long-term debt holders that get their payment $F_2$ first (they are senior to any new financings), new financiers get repaid only when $X^G$ realizes with prob. $1/2$. This reflects the standard debt overhang in Myers (1977).

The firm needs to use this refinancing capacity to pay the maturing short-term debt $F_1$. We focus on the case that $X^G > 2F_1 + F_2$, so that short-term debt can get refinanced at $t=1$ even without investment. In other words, the short-term debt is riskless, and therefore imposes no overhang. Note that this condition implies that in state $Gu$ (with asset-in-place value $1.5X^G$) not only the short-term debt is riskless, but also the long-term debt is riskless (as $X^G > F_2$).

To determine the firm’s investment incentives at $t=0$, we calculate the value increment of long-term debt due to new investment. Without investment, the long-term debt date-$0$ value is $0.75 \times F_2$. With investment, the long-term debt value becomes $\frac{3}{4} \times F_2 + \frac{1}{4} \times \min(Y^G, F_2)$. Therefore, the firm will invest if and only if

$$\frac{1}{4} \min(Y^G, F_2) < \gamma^G$$

We assume throughout that $\frac{1}{4} Y^G > \gamma^G$; otherwise the firm invests always regardless of $F_2$. Therefore, the firm will invest if and only if $F_2 < 4\gamma^G$, which is the standard debt overhang of Myers (1977).

3.2.2 Bad State B

\textbf{Without investment.} When bad news about the firm’s asset-in-place hits, $X^B$ may be sufficiently low that the firm cannot successfully refinance the short-term debt at state $Bd$. Formally, we assume that

$$X^B < 2F_1 + F_2,$$

\textsuperscript{12} This differs from Diamond (1991), where it is the manager’s non-pleageable control rent combined with the need for outside funding drives the inefficiency. In Diamond (1991), if the manager had deep pockets to refinance the firm, then he would internalize the loss of non-pleageable control rent and eliminate the inefficiency. In this model without control rent, there is no distinction between existing equity holders/managers or outside equity holders.
and the firm defaults on its short-term debt at \( t=1 \). Therefore, without investment, the total debt value (the sum of short-term and long-term) at state \( Bd \) at \( t=1 \) is \( \frac{X^B}{2} \), which occurs with probability 0.5.

Now we study state \( Bu \). Because we are mainly interested in preserving the first best investment policy, and since investment NPV is moderate, we focus on the situation that \( 1.5X^B > F_1 + F_2 \). This implies that at state \( Bu \) both debts are riskless, and as a result equity holders will recover all investment benefit there. Therefore, without investment the total debt value at state \( Bu \) is \( F_1 + F_2 \).

In sum, without investment the total debt value at \( t=0 \) is

\[
0.5 \times (F_1 + F_2) + 0.5 \times \frac{X^B}{2}
\]

With investment. Now we calculate the total debt value given investment. The next lemma shows that in order for the firm to invest at \( t=0 \), it must be the case that the firm with investment can successfully refinance the short-term debt at state \( Bd \).

**Lemma 1.** Assume that \( \gamma^B < \frac{1}{2} Y^B \). Then the necessary condition for the firm to invest at \( t=0 \) is successful refinancing short-term debt at state \( Bd \).

**Proof.** Suppose not. At \( Bd \) the total debt value is the firm value \( \frac{X^B}{2} + Y^B \), and at \( Bu \) the debt holders get paid in full. Relative to the situation without investment, this implies that standing at \( t=0 \) the debt holders gain by \( 0.5 \times Y^B \). Given that \( \gamma^B < \frac{1}{2} Y^B \), equity holders cannot break even. QED.

Given this lemma, we only need to discuss two cases depending on whether the long-term debt is risky or not.

**Case 1.** When \( Y^B \geq F_2 \) so long-term debt is paid in full in state \( Bdd \). This implies that both short-term debt and long-term debt become riskless, and the total debt value at \( t=0 \) is \( F_1 + F_2 \). Therefore the value increment due to new investment is \( \frac{1}{2}(F_1 + F_2) - \frac{1}{4}X^B \). Comparing to positive NPV \( \gamma^B \), the necessary condition for investment is
Figure 5: \((F_1, F_2)\) space with different investment policy and \(t=1\) debt values \(D^{-1}\) in the model.

**Case 2.** When \(Y_B < F_2\) so the long-term debt only recovers \(Y_B\) at the state \(Bdd\). The total debt value given investment at \(t=0\) becomes \(F_1 + \frac{3}{4}F_2 + \frac{1}{4}Y_B\). Therefore, relative to (1), the debt value increment due to new investment is \(\frac{1}{2}\left(F_1 + \frac{F_2}{2}\right) + \frac{Y_B}{4} - \frac{X_B}{4},\) and the necessary condition for investment is

\[
\frac{1}{2}\left(F_1 + \frac{F_2}{2}\right) + \frac{Y_B}{4} - \frac{X_B}{4} \leq Y_B \iff F_1 + \frac{F_2}{2} \leq 2Y_B - \frac{X_B}{2}.
\]

In this case the new investment bails out short-term debt, but cannot make long-term debt holders to be paid in full. Interestingly, the total debt value increment is \(\frac{1}{2}\left(F_1 + \frac{F_2}{2}\right) + \frac{Y_B}{4} - \frac{X_B}{4} ,\) where the short-term face value \(F_1\) and the long-term face value \(F_2\) receive different weights. To
be specific, the short-term debt has a greater weight, which is 1, than the long-term debt has, which is $\frac{1}{2}$. This difference is the underlying reason why short-term debt holders are able to extract more value increment from new investment than long-term debt holders. Essentially, the weight difference reflects the fact that at state $Bd$, the short-term debt gets paid in full, while for long-term debt the firm pays it in full only when state $Bdd$ occurs (with prob. 0.5).

### 3.3 Model Solution

The firm is choosing debt maturity $(F_1, F_2)$ at $t=-1$ to minimize overhang at $t=0$. In the following analysis, we are mainly interested in characterizing the optimal debt maturity structure that achieves the maximum $t=-1$ target debt value $D_{\text{target}}$ while preserving the first-best investment policies at both states at $t=0$. We illustrate the firm’s investment decisions on the space of debt structure $(F_1, F_2)$ in Figure 5. As the firm tries to meet the target debt value $D_{\text{target}}$ at $t=-1$, we also calculate the total debt value $D^{-1}$ at $t=-1$ for all regions.

We have shown that raising $F_2$ and reducing $F_1$ at $t=-1$ is a good idea from the stand point of state B. However, the analysis in state G implies that the firm cannot raise $F_2$ too much: there, the riskless short-term debt is harmless, and it is only $F_2$ that imposes overhang effect. This trade-off determines the optimal interior debt maturity that we are after.

In Figure 5, only region R1 achieves the first best investment in both states. There are two sub-regions, depending on whether $F_2 > Y^B$, i.e., whether long-term debt given investment is riskless or not. This is also the kink point shown in Figure 5. According to results after Lemma 1 in Section 3.2.2, when $F_2 \leq Y^B$, both long-term and short-term debt are risk free, with $t=-1$ value

$$D^{-1} = F_1 + F_2,$$

and the boundary line is $F_1 + F_2 = 2Y^B + \frac{X^B}{2}$. When $F_2 > Y^B$, long-term debt only defaults at the state $Bdd$, and $D^{-1} = F_1 + \frac{3F_2 + Y^B}{4}$, with the boundary line

$$F_1 + \frac{F_2}{2} = 2Y^B - \frac{Y^B}{2} + \frac{X^B}{2}.$$  

---

Note that in Figure 5 we have implicitly assumed that $4Y^G > Y^B$. This condition also requires that $Y^G > Y^B$ (more precisely, the investment payoff at state $G$ is greater than that in state $B$) given our earlier assumption $4Y^G > Y^G$ in Section 3.2.1. This assumption allows for the case that the state-G first-best long-term face $F_2 = 4Y^G$ exceeds the...
From Figure 5, we see that the optimal debt maturity structure that achieves the first-best investment policies and the maximum t=-1 target debt value $D_{\text{target}}$ is the intersection point between $F_2 = 4\gamma^G$ and $F_1 + \frac{F_2}{2} = 2\gamma^B - \frac{Y^B}{2} + \frac{X^B}{2}$, i.e., (the asterisk in Figure 5)

$$F_1^* = \frac{X^B}{2} + 2\gamma^B - \frac{Y^B}{2} - 2\gamma^G, F_2^* = 4\gamma^G.$$

With this debt maturity structure, the maximum t=-1 debt value that preserves the optimal investment policies is

$$D_{\text{target},*} = \frac{X^B}{2} + 2\gamma^B + \gamma^G - \frac{Y^B}{4}.$$

We briefly discuss the situation where the firm’s date -1 target debt level $D_{\text{target}} > D_{\text{target},*}$, so that the firm is forced to choose some point in the non-first-best region. Here the trade-off between different states kicks in. If $\gamma^B > \gamma^G$, then the region R2 is chosen where relatively more long-term debt is used to maximize the investment incentives at state B. On the other hand, if $\gamma^B < \gamma^G$, then R3 is chosen where the firm takes relatively more short-term debt to maximize the investment incentives in state G. As our main objective of this paper is to show the trade-off between short-term and long-term debt, the detailed characterization is less interesting.

4. Extensions and Discussions

4.1 Endogenous Costly Default

Costly endogenous default, the extreme version of underinvestment, is just one symptom of short-term debt overhang. Endogenous default has been analyzed in Bulow and Shoven (1978), Black and Cox (1976), and Leland (1994). For instance, in Leland and Toft (1996) and He and Xiong (2010), equity holders, facing a low firm fundamental value, might find suboptimal to keep absorbing the financial losses in rolling over maturing debt. As a result, equity holders default, leaving debt holders to bear the bankruptcy cost. Essentially, it is because equity holders do not want to subsidize debt holders, especially the maturing ones.

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state-B lower bound cash flows $Y^B$. As a result, even with investment, in state B the long-term debt with $F_2 = 4\gamma^G$ can still be risky at t=2 given the worst cash flow realization, while short-term debt which gets paid in t=1 becomes riskless. There are other ways, e.g., random investment opportunities (symmetric to both states), to achieve this goal, but we deem that the unnecessary modeling complicity outweighs the potential benefit.

14 The interpretation of endogenous default given debt burden as “underinvestment” due to debt-overhang, is mentioned in, for example, Lambrecht and Myers (2008) and He (2009).
To see the equivalence, let us examine the endogenous default decision at t=1 state $d$ in this model. Ignore the investment decision at t=0, and assume a zero liquidation value of the firm. We can interpret it as that bankruptcy involves a substantial dead-weight loss, which essentially implies that default represents severe underinvestment. As shown in Section 3.2, the new financiers can at most recover $\frac{X - F_2}{2}$ from the date two cash flows (suppose that the asset-in-place $X$ at state $du$ is above $F_2$), and therefore are not willing to refinance the maturing short-term debt if and only if

$$\frac{X}{2} < F_1 + \frac{F_2}{2}.$$  

When this condition holds, the firm inefficiently defaults, leaving both long-term and short-term debt worthless. This inefficient default is neither because the firm cannot get fairly priced outside-financing (due to informational problems or financial market disruption), nor because the firm has some non-pledgeable part of future cash flows (a la Diamond (1991)) that new financiers cannot internalize. Rather, it is because in order to bail out the firm, the firm/new financiers need to repay its short-term debt fully, and also subsidize the long-term debt holders with $\frac{F_2}{2}$. This reflects debt overhang, as the inefficiency is rooted in the fact there cannot be renegotiations between existing debt holders who demand payment (either immediate as short-term debt, or future as long-term debt), and the firm who makes the default decision.

The interesting point regarding endogenous default is that, when the overhang effect is about failing to attract new financings to avoid firm’s inefficient early default, it is the maturing short-term debt which demands immediate repayment that plays a more significant role. This point is clearly reflected in the bankruptcy threshold $F_1 + \frac{F_2}{2}$, which puts a greater weight on short-term debt. The intuition is that the short-term debt gets paid sooner than long-term debt, and there are more uncertainties resolved (especially good shocks) in the future to reduce the long-term overhang. More specifically, in bailing out the firm from default, new financiers pay the maturing short-term debt in full, subsidizing them one-to-one. However, by keeping the firm alive the wealth transfer from new financiers to long-term debt holders is typically less than one-to-one (in our example it is the probability $\frac{1}{2}$ of the $du$ realization at t=2), because it is possible for the firm to escape the default region tomorrow.
4.2 Credit Lines (Revolvers) and Market Based Pricing

In practice, many firms have standing credit lines (also called revolving lines of credit or revolvers) issued by banks. The mechanism of credit line works as an insurance contract: The firm typically pays a fee up front to secure the line, and will later draw down the line if the precommitted rate is below the market one. During the 2007/2008 crisis, the credit line draw-downs accounted for the major part of loans extended by commercial banks (Ivashina and Scharfstein (2009)).

Existing theory about credit lines emphasizes on helping firms overcome liquidity shocks or alleviate risk shifting incentives (e.g., Boot and Thakor (1994)). Our model suggests that credit lines also alleviate short-term overhang by fixing the refinancing cost in bad times. We provide two discussions regarding the debt overhang, credit line, and its recent market-based pricing.

4.2.1 Can the seniority of credit line resolve debt overhang?

In practice, the credit lines/revolvers from banks have the distinct feature that draw-downs are often senior to any existing debt. One tends to think that this seniority of credit line should resolve the debt overhang problem completely, because it directly attacks the heart of debt overhang: Banks who issued revolvers do not have to worry that the first dollar out of the new investment goes to existing senior debt holders.

There is a subtle but important difference between the incentives of new financiers (banks in the credit line case) and the incentives of the firm who decides whether to draw down the line. In fact, because banks are already obligated to provide financing if the firm decides to do so, the seniority of drawdowns plays no role at all in the firm’s investment incentives. The only thing that matters in this scenario is the pricing of drawdowns. Given the relatively expensive drawdowns which get to be repaid first later on, the firm may decide not to draw the line if almost all the future investment benefit goes to the bank.\textsuperscript{15} For illustration, consider the following extreme example with zero risk-free rate. Because revolvers needs to break even when they are issued, usually banks set a future drawdown rate \( r > 0 \) higher than the risk-free rate. Clearly, the firm will decide not to draw the line for the positive NPV projects that yields constant returns below \( r \).

\textsuperscript{15} Of course, the ex post seniority helps the bank set a low drawdown rate ex ante, which alleviates overhang.
4.2.2 Market based pricing

A recent innovation to the contract of credit-lines/revolvers is market-based pricing; that is, the interest rate of new-drawdowns is partially tied to the firm’s current strength. This is a form of performance-based pricing which is common in the bank debt (e.g., Asquith, Beatty, and Weber (2005)). More specifically, when the firm draws on the remaining line, the drawn spread is positively tied to its credit default swap (CDS) spread.\(^\text{16}\)

The repricing of the cost of borrowing against the line leaves the firm’s cost of borrowing higher when the firm’s prospects are bad, much like short-term debt.\(^\text{17}\) To the extreme case where the drawdown rate is fully market based, then this essentially takes away the insurance that the bank offered to the firm. As a result, there is no difference between new financiers and the bank, and we are back to the model we have analyzed before.

However, in our model, debt overhang is not all about insurance. There is a key difference between endogenous default decision and the investment/maintenance decision. Insurance, by forcing a subsidy from bank to maturing debt holders, always alleviates the distortion of default decision. However, for investment, the heart of overhang lies in the firm’s “incentives,” which can be positively related to market-based pricing.

This point suggests that a properly designed market-based pricing on credit lines is better than credit lines without market-based-pricing. The firm knows that its investment may improve its long-term debt CDS, and this will in turn reduces the firm’s future financing cost. As a result, market-based pricing provides the firm extra incentives to invest at t=0. Essentially, a properly designed market-based-pricing should combine two components. The first is the insurance that protects the firm from those states with deteriorating asset-in-place, and the second is the performance-based sensitivity that entices the firm’s investment. Therefore, the core idea here is similar to optimal contracting with moral hazard, which is to reward/punish the firm for its actions (here, maintenance) but not for fundamental states beyond its actions.

Interestingly, this further suggests that although long-term debt CDS has its own advantages (e.g., more liquid and accurate pricing) over short-term debt CDS to be the market-base in

\(^{16}\) Typically these revolvers with market-based-pricing specify floor and cap which are the minimum and maximum of the drawn spreads. If the benchmark CDS spread is lower than the floor, the floor applies for the drawn margin. Conversely, if the CDS spread at the time of the draw is higher than the cap, the cap is applied. Finally, if the CDS spread at the time of the draw falls in between the floor and cap, certain formula will apply. Source: Thomson Reuters LPC, Markit on Reuters 3000 Xtra / Credit Views.

\(^{17}\) For a rigorous study of debt with performance-based-repricing and its implications on default, see Manso, Strulovici, Tshistyi (2010).
designing the market-based-pricing scheme, the ideal market-base-measure would be state-dependent. In bad times, stronger short-term overhang implies that short-term debt CDS might contain more action-based information than long-term debt CDS does, simply because the wealth transfer due to investment is greater for short-term debt when the firm is close to default.

4.3 Endogenizing Leverage \( D_{\text{target}} \) at \( t=-1 \)

Now we endogenize the target date-0 debt value. A tradeoff between saving taxes versus increasing bankruptcy costs will yield a positive value of \( D_{\text{target}} \), for standard reasons. There is not much special about this approach in our framework. An alternative is to account for the control role of debt, and of short-term debt in particular. This section describes such an approach. This added structure will prove to be useful in discussion of cash reserve in Section 4.4.

Suppose that at \( t=-1 \) the entrepreneur owns the patent of the project. To start this project, the entrepreneur with no personal wealth needs 1 dollar of initial investment. He can raise this initial investment through debt or (outside) equity, and becomes the manager of the firm afterwards. We assume that equity holders are soft claims that are subject to renegotiation.

We will use a very simple model motivated by Jensen (1986), Hart and Moore (1994), Diamond and Rajan (1999), and Diamond (2004, 2006). It introduces managerial “equity overhang,” where managers take a fraction \( \lambda \) of all free cash flow in excess of debt payments. We follow Diamond (2004, 2006) and assume that default on debt allows the legal system to prevent the manager from consuming any cash they divert instead of paying out or investing (giving the manager nothing if the legal sanction is imposed). Legal sanctions remove any of the benefit of diversion which occurs that period, but not the benefit of diversion in previous periods. Contracts are written such that this legal sanction is imposed if a debt contract is not paid on its due date. The threat of this legal sanction ensures that debt is paid when cash is available, and because debt cannot be renegotiated, the default automatically imposes the sanction. However equity contracts (which are soft) do not have automatic sanctions so that equity holders have no right to impose the legal sanction for default. We simply assume that the manager can take a fraction \( \lambda \) of remaining cash flows, which can be motivated by that the manager is able to directly divert all free cash and retain a fraction \( \lambda \) of it (while destroying a

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18 This model is a much simplified of that in Diamond (2006). We simplify by assuming that no debt can be renegotiated and that the legal sanctions of debt default completely eliminate proceeds from managerial diversion on the date when default occurs.
fraction $1 - \lambda$) in a way that cannot be verified or recovered without legal sanctions. As a result, owners of outside equity allow the manager to take a fraction $\lambda$ of current free cash flow each period, if the manager so desires, given its effect on his current or future payoffs.\footnote{So essentially we are modeling outside equity in this setup. See Myers (2000) for another way of modeling.}

To recap, at $t=-1$ the firm raises equity and debt to carry out the initial investment.\footnote{We assume that the manager can commit to invest funds properly on the instant they are received, but cannot commit not to divert cash flows obtained in excess of immediate investment needs or cash flows obtained from the returns to investments.} At $t=0$ the firm requires new investment as analyzed in Section 3, and at $t=1$ the refinancing decision is controlled by the existing shareholders. Because issuing any new equity only benefits the entrepreneur and dilutes their own value, the new financing is in the form of (junior) debt. And $t=2$ the manager can get (at least) $\lambda$ fraction of free cash flows after the debt payment.

Denote by $M_{-1}$ the value of entrepreneur/manager, $E_{-1}$ the value of (outside) equity, and $D_{-1}$ the value of total long-term and short-term debt, all evaluated at $t=-1$. The agency problem at $t=2$ implies that $M_{-1} \geq \frac{\lambda}{1-\lambda} E_{-1}$, i.e., the manager has to have sufficient inside stake for him to behave. The initial investment requires that $E_{-1} + D_{-1} \geq 1$; if this inequality holds strictly then the entrepreneur/manager can consume the difference at date -1. Therefore the entrepreneur’s date -1 value is $M_{-1} + (E_{-1} + D_{-1} - 1)$. Finally, denote by $v(D_{-1})$ the firm value as a function of $D_{-1}$ (which is an decreasing function) where the firm value is determined by the optimal maturity structure determined in Section 3. Finally, the accounting identity implies that $M_{-1} + E_{-1} + D_{-1} = v(D_{-1})$. Therefore, the manager who chooses the $t=-1$ financial structure solves the following problem:

$$\max M_{-1} + (E_{-1} + D_{-1} - 1)$$

s.t. $E_{-1} + D_{-1} \geq 1$, $M_{-1} \geq \frac{\lambda}{1-\lambda} E_{-1}$, $E_{-1} + M_{-1} + D_{-1} = v(D_{-1})$

**Proposition 2.** Assume that $(1-\lambda)v(0) < 1$ and $v(1) > 1$. Then the optimal date-$0$ debt value $D_{\text{target}} \in (0,1)$ is the smallest solution to the equation $D_{-1} + (1-\lambda)(v(D_{-1}) - D_{-1}) = 1$. 

The first restriction \((1 - \lambda) v(0) < 1\) implies that using outside equity only cannot raise enough capital to cover initial investment; and the second condition \(v(1) > 1\) implies that it is feasible to raise the entire investment capital by debt. Then the optimal date-0 debt will be an interior solution. Finally, the equation in Proposition 1 simply says that the debt holders and outside equity holders (who have a \(1 - \lambda\) fraction of total equity value \(v(D_{-1}) - D_{-1}\)) contribute the entire initial investment 1 (as the manager has zero initial wealth.)

4.4 Cash Reserve?

One potential solution to the debt overhang is that the firm maintains cash reserves. We will first investigate the role of cash reserve by ignoring the agency issue of managerial diverting that we introduced in Section 4.3, then discuss the interesting interaction between the managerial diverting and debt overhang.

4.4.1 State-contingent maturity and callable long-term debt

It is clear that raising cash reserves that are not subject to agency problems (diverting, dividend payout, etc.), while holding total debt issuance fixed, could alleviate debt-overhang—simply because we can interpreted cash as negative debt. The more interesting question is, can the firm reduce overhang by issuing more debt at \(t=-1\), say \(D_{-1} + C\) where \(C > 0\), and keeping \(C\) inside the firm as cash reserve? The answer is yes.

In our model, if the firm can issue debt with state-contingent maturity, then the optimal contract will be short-term debt in state G (so there is no overhang) and long-term debt in state B (so there is less overhang). This result indicates that the callable feature of long-term debt can help on this dimension, because the firm who issued callable long-term debt at \(t=-1\) can choose to call these debt at \(t=1\) if state G realizes.\(^{21}\) Of course, in order to motivate the firm to call back the debt in full, the call price should be at a discount, i.e., below the long-term debt value given investment. Otherwise, the same extent of wealth transfer suggests that the firm will decide to do nothing. Also, a pre-determined call price cannot deal with random investment opportunities.

Interestingly, the state-contingent callable feature can also be generated by a cash reserve. It is because cash allows the firm to have a state-contingent repayment policy. Specifically, in bad

\(^{21}\) Bodie and Taggart (1978) suggest that callable long-term debt can alleviate overhang. However, Bodie and Taggart (1978) still deem it as a puzzle why firms do not simply roll over their short-term debt, which suggests that the authors do not realize that roll over short-term debt can impose stronger overhang in some states.
states, the firm can use the cash to pay part of short-term debt at $t=1$, while in good states the firm will save these cash and use them to pay part of long-term debt at $t=2$. Therefore, this state-contingent repayment policy help the firm transform some long-term (short-term) debt to short-term (long-term) debt in state G (B), which is value improving in this model.

4.4.2 Debt overhang on manager’s diverting decision

Of course, the very reason to have debt in the first place, as discussed in Section 4.3, is because the manager can divert some of the firm’s free cash flows. This militates against having the firm pile up extra cash, because for any dollar that sitting in the firm at $t=1$ in excess of short-term debt payment, the manager can divert it to obtain $\lambda$ at $t=1$ (and the date 2 default is too late to recover it).\(^{22}\) Interestingly, different from the standard argument (e.g., DeMarzo and Fishman (2007)) that the manager’s inside equity stake $\lambda$ will prevent him from diverting the cash at the interim date $t=1$ (because he can get $\lambda$ fraction of free cash flows at $t=2$), in our setting the manager will strictly prefer to divert at $t=1$, if the outside equity holders cannot promise to the manager more than $\lambda$ fraction of $t=2$ free cash flows. The reason is just debt overhang: the manager understands that the cash left in the firm goes to the debt holders first at $t=2$, and therefore he has a strict incentive to divert at today rather than wait to share the cash tomorrow.

For illustration, suppose that there is one dollar of free cash flow in the firm at $t=1$, and in our model at $t=2$ the firm is solvent with probability $\frac{1}{2}$. Then diverting today gives the manager a utility of $\lambda$, while the expected value by waiting to share the free cash flows with outside equity holders at $t=2$ is only $\lambda/2$. Here, the manager will divert earlier along the equilibrium path because the debt coming due tomorrow hangs over his own “inside-equity.” As a result, holding extra cash does not overcome the short-term debt overhang of injecting new cash because the manager’s payoff is very much like that of newly-injected equity.

5. Conclusion

Debt overhang influences the investment and default decisions of those whose claims are like equity. Long term debt causes overhang, because it prevents equity from receiving any payoff from investment when the ex-post payoff is low enough that there is a default. Short-term debt with the possibility of default can impose even greater overhang, simply because there is less uncertainty resolved over the shorter time until it matures, and as a result most of the first

\(^{22}\) Here, for the sake of argument, we only consider the manager’s diverting incentives at $t=1$. 

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part of any initial increase in value (due to investment or bailout to avoid default) will not result in any payoff to equity. Short-term debt then imposes either no overhang (if riskless) or large overhang (if likely to default). The timing of debt maturity can have a major impact on investments, especially on investments that can help avoid default.

The problems caused by large impending debt maturity go beyond the risk of runs and limited access to liquidity. The timing of repayments, access to lines of credit, and the pricing of credit lines all combine to either amplify or reduce the risks of potential default.

Appendix

A.1 Appendix for Section 2.3.2

A.1.2 Proof for Proposition 1

Consider $V_2 = V_0 \exp \left( \bar{X}_1 - \frac{1}{2} + \bar{X}_2 - \frac{\sigma^2}{2} \right)$, where $Q = 0$ and

$$\tilde{\sigma}_2 = \begin{cases} 0 & \text{when } \bar{X}_1 > 0 \\ \sigma = \varepsilon & \text{when } \bar{X}_1 \leq 0 \end{cases}$$

where $\varepsilon$ is sufficiently small. When $\sigma = 0$, the second period adds no risk, and as a result long-term debt is identical to short-term debt. As a result, both debts have the same value and overhang. We will set $\tilde{F}_2 = \exp(-0.5)$, so that potential second period noise occurs exactly when the long-term debt is at the money. As we have seen in the discussion, this gives the best chance to have smaller long-term overhang.

Our road map is as follows. We consider a perturbation from $\sigma = 0$ to $\sigma = \varepsilon > 0$, and comparing the change of overhang effects on each debt. Of course we need to control for debt value, and we will adjust $F_1$ (which is much easier) to ensure the same debt value. Then we can check the following object (where we denote debt overhang by $OH_t = 1 - \Delta_t$):

$$\frac{dOH_1}{d\sigma} - \frac{dOH_2}{d\sigma} = \frac{dOH_1}{dF_1} \frac{dF_1}{d\sigma} - \frac{dOH_2}{dF_1} \frac{dD_1}{d\sigma} - \frac{dOH_2}{dD_1} \frac{dF_1}{d\sigma}$$

(2)

If it is positive, then we prove our claim, as when $\sigma = 0$ both debts have the same overhang.
We will show that raising $\sigma$ from 0 has no first order effect on long-term debt value $D_2$, i.e.,
\[
\frac{dD_2}{d\sigma}_{\sigma=0} = 0 \quad \text{while} \quad \frac{dOH_2}{d\sigma}_{\sigma=0} < 0.
\]
Therefore, since $dD_1/dF_1 > 0$, we obtain our result. To show these results, we have (note $Q = \ln F_2 + 0.5$)
\[
D_2(\sigma) = F_2 \int_{-\infty}^{\infty} n(x) dx + \int_{-\infty}^{0} \left[ \int_{-\infty}^{-x+\frac{\sigma^2}{2}} \exp\left(x - \frac{1+\sigma^2}{2} + y\right) n\left(\frac{y}{\sigma}\right) \frac{1}{\sigma} dy + F_2 \int_{y=\frac{\sigma^2}{2}}^{\infty} n\left(\frac{y}{\sigma}\right) \frac{1}{\sigma} dy \right] n(x) dx
\]
\[
= F_2 \int_{-\infty}^{\infty} n(x) dx + F_2 \int_{-\infty}^{0} \left\{ \exp(x) \int_{-\infty}^{-x+\frac{\sigma^2}{2}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-\sigma^2)^2}{2\sigma^2}\right) dy + \int_{-\infty}^{\infty} n\left(\frac{y}{\sigma}\right) \frac{1}{\sigma} dy \right\} n(x) dx
\]
let $t = \frac{y-\sigma^2}{\sigma^2}$
\[
\quad = \sigma F_2 \int_{0}^{\sigma} n(x) dx + F_2 \int_{-\infty}^{0} \left\{ \exp(x) \int_{-\infty}^{-x+\frac{\sigma^2}{2}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-\sigma^2)^2}{2\sigma^2}\right) dy + \int_{-\infty}^{\infty} n\left(\frac{y}{\sigma}\right) \frac{1}{\sigma} dy \right\} n(x) dx
\]
Therefore
\[
\frac{dD_2(\sigma)}{d\sigma} = F_2 \int_{-\infty}^{0} \left\{ \exp(x) n\left(\frac{-x}{\sigma} - \frac{\sigma}{2}\right) \left(\frac{x}{\sigma^2} - \frac{1}{2}\right) - n\left(\frac{-x}{\sigma} + \frac{\sigma}{2}\right) \left(\frac{x}{\sigma^2} + \frac{1}{2}\right) \right\} n(x) dx
\]
\[
= F_2 \int_{-\infty}^{0} \left\{ n\left(\frac{-x}{\sigma} + \frac{\sigma}{2}\right) \left(\frac{x}{\sigma^2} - \frac{1}{2}\right) - n\left(\frac{-x}{\sigma} + \frac{\sigma}{2}\right) \left(\frac{x}{\sigma^2} + \frac{1}{2}\right) \right\} n(x) dx
\]
\[
= -F_2 \int_{-\infty}^{0} n\left(\frac{-x}{\sigma} + \frac{\sigma}{2}\right) n(x) dx
\]
\[
= -\sigma F_2 \int_{-\infty}^{0} n\left(\frac{-u+\sigma}{2}\right) n(\sigma u) du
\]
Which is zero when $\sigma = 0$. However, since
\[
OH_2(\sigma) = 1 - \int_{-\infty}^{\infty} n(x) dx - \int_{-\infty}^{-\frac{x+\sigma^2}{2}} \frac{1}{\sigma} dy n(x) dx = \int_{-\infty}^{\infty} n(x) dx - \int_{-\infty}^{\int_{-\infty}^{\infty} \frac{\sigma}{2} n(t) dt n(x) dx
\]
And the first order effect on overhang by raising $\sigma$ is
\[
\frac{dOH_2(\sigma)}{d\sigma} = \int_{-\infty}^{0} n\left(\frac{-x}{\sigma} + \frac{\sigma}{2}\right) \left(\frac{x}{\sigma^2} + \frac{1}{2}\right) n(x) dx = \int_{-\infty}^{0} n\left(\frac{-x}{\sigma} + \frac{\sigma}{2}\right) \left(\frac{x}{\sigma^2} + \frac{1}{2}\right) n(x) dx
\]
\[
\text{let } t = \frac{x+\sigma^2}{2}
\]
\[
= \int_{-\infty}^{t} n\left(-t + \sigma\right) \text{ln} \left(\sigma - \frac{\sigma^2}{2}\right) dt
\]
\[
\text{let } \sigma = 0
\]
\[
= - \frac{1}{\sqrt{2\pi}} < 0.
\]
Therefore we proved our claim. QED.
A.1.2 Calculating long-term debt value and long-term overhang

The calculation of long-term debt value and its overhang involves a two-dimensional integral. Here we show that one can transform them into one-dimensional integrals, which allow for much more accurate numerical evaluation.

Let \( Y_2 = \tilde{X}_1 - \frac{\sigma_i^2}{2} + \tilde{X}_2 - \frac{\sigma_i^2}{2} \), so the firm value at \( t=2 \) is \( \exp(Y_2) \). Its probability density is

\[
f(y) = \int_{-\infty}^{\infty} n \left( \frac{y + \frac{\sigma_i^2}{2} + \frac{\sigma_H^2}{2} - x}{\sigma_H \sigma_1} \right) n \left( \frac{x}{\sigma_1} \right) dx + \int_{-\infty}^{\infty} n \left( \frac{y + \frac{\sigma_i^2}{2} + \frac{\sigma_L^2}{2} - x}{\sigma_L \sigma_1} \right) n \left( \frac{x}{\sigma_1} \right) dx.
\]

The long-term debt value is \( D_2 = \int_{-\infty}^{\infty} \exp(y) f(y) dy + F_2 \int_{-\infty}^{\infty} f(y) dy \) which involves a two-dimensional integral. However, we easily reduce one dimension. Take the first term as an example. We have

\[
\int_{-\infty}^{\infty} \exp(y) \int_{-\infty}^{\infty} n \left( \frac{y + \frac{\sigma_i^2}{2} + \frac{\sigma_H^2}{2} - x}{\sigma_H \sigma_1} \right) n \left( \frac{x}{\sigma_1} \right) dx dy + \int_{-\infty}^{\infty} \exp(y) \int_{-\infty}^{\infty} n \left( \frac{y + \frac{\sigma_i^2}{2} + \frac{\sigma_L^2}{2} - x}{\sigma_L \sigma_1} \right) n \left( \frac{x}{\sigma_1} \right) dx dy
\]

and the first term here can be written as

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(y) n \left( \frac{y + \frac{\sigma_i^2}{2} + \frac{\sigma_H^2}{2} - x}{\sigma_H \sigma_1} \right) n \left( \frac{x}{\sigma_1} \right) dx dy
\]

\[
= \int_{-\infty}^{\infty} \exp \left( \frac{1}{2\sigma_H^2 \sigma_1^2} \left( 2\sigma_H^2 \sigma_1 y - \sigma_1^2 \left( y + \frac{\sigma_i^2 + \sigma_H^2}{2} \right) \right) \right) \int_{-\infty}^{\infty} \exp \left( \frac{1}{2\sigma_H^2 \sigma_1^2} \left( 2\sigma_H^2 \sigma_1 x - \sigma_1^2 \left( y + \frac{\sigma_i^2 + \sigma_H^2}{2} \right) \right) \right) dx dy
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_H \sigma_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_1} \exp \left( - \frac{1}{2} \frac{\sigma_1^2 \left( y + \frac{\sigma_i^2 + \sigma_H^2}{2} \right)^2}{\sigma_i^2 + \sigma_H^2} \right) dx dy
\]
Let \( t = \frac{\sqrt{\sigma_1^2 + \sigma_H^2} \left( x - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_H^2} \left( y + \frac{\sigma_1^2 + \sigma_H^2}{2} \right) \right)}{\sigma_H \sigma_1} \), then this term becomes

\[
\int_{-\infty}^{\ln F_2} \frac{\exp \left( -\frac{1}{2} \left( \frac{y - \sigma_1^2 + \sigma_H^2}{\sigma_1^2 + \sigma_H^2} \right)^2 \right) \left( \frac{\sigma_1^2}{\sigma_1^2 + \sigma_H^2} \left( y + \frac{\sigma_1^2 + \sigma_H^2}{2} \right) \right)}{\sqrt{2\pi(\sigma_1^2 + \sigma_H^2)}} dy
\]

The other terms can be calculated similarly. Then we use Matlab built-in function *quadgk.m* to numerically evaluate the one-dimensional integrals, with the accuracy level at 1e-12.

**A.2 Proof of Proposition 2**

The manager’s optimization problem is equivalent to \( \max v(D) \) s.t. \( E + D \geq 1 \), and \( M \geq \frac{\lambda}{1 - \lambda} E \), where we use \( D, E, M \) to indicate their \( t=-1 \) value. Because \( v(D) \) is decreasing in \( D \), the first constraint is binding; otherwise lowering \( D \) improves. Then the second constraint is binding as well, because otherwise a higher \( M \) raises \( E \) which in turn reduces \( D \). As a result, we have that

\[
\frac{1}{1 - \lambda} E + D = v(D) \Rightarrow D + (1 - \lambda) (v(D) - D) = 1.
\]

Let \( Q(D) = D + (1 - \lambda) (v(D) - D) \). Under our assumptions, \( Q(1) > 1 \) and \( Q(0) < 1 \). Also, as we have seen in Figure 5, if there is any discontinuity on \( v(D) \), it only jumps downward where certain efficient investment starting to be cut. Hence, \( Q(D) \) only jumps downward, and there always exists a solution \( D \in (0, 1) \) to the above equation (take the smallest one if multiple solutions exist). Q.E.D.

**Reference**


