Dynamic Compensation Contracts with Private Savings

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Abstract

This paper studies a dynamic agency problem where a risk-averse agent can privately save. In the optimal contract, wages are downward rigid; permanent pay raises occur when the agent’s historical performance is sufficiently good; and when the agent is dismissed due to his poor performance, he walks away with a severance pay to support his post-firing consumption at the current wage level. Thus, under realistic contracting frictions, seemingly inefficient compensation schemes can indeed be optimal. Several extensions are considered, including the agent’s outside option and renegotiation-proof contracts.

Key Words: Continuous-time Contracting, Poisson Process, Wealth Effect, Cost of High-Powered Incentives.

JEL Codes: D86, J31 J33.

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1 Introduction

This paper studies a dynamic agency problem where the risk-averse agent controls the firm’s profitability through unobservable actions, and he can save privately (or, secretly save, hidden savings). We find that in the optimal compensation contract, wages are non-decreasing, and the agent gets a severance pay even when he is dismissed due to his poor performance. Both features are commonly observed in reality.

In the dynamic contracting literature, Rogerson (1985) among others has found that without private savings, the optimal wage pattern tends to be “front-loaded,” which means that the agent’s expected marginal utility from consumption increases over time—the so-called inverse-martingale property. With access to a private savings account, the agent will smooth his consumption, thereby devastating the incentive scheme designed in the front-loaded contract. The optimal contract derived in this paper features a back-loaded, non-decreasing wage pattern, and the agent’s motive in private saving is absent.

The general optimal contracting problem with private savings is complicated; we solve this model under a specific setting. In this paper, cash flows follow a Poisson process. The cash flow arrival intensity is controlled by the agent’s three levels of unobservable effort (action): *shirking*, *working*, and *myopic*, and the optimal contract implements the interior working effort. Shirking leads to no cash flow in the next time interval, and working generates a positive success intensity. The myopic action helps to improve the short-term “hard” cash flow performance, but hurts the firm’s long-run value. We envision that these long-run destruction, usually taking forms unforeseen by investors, will be realized after the agent’s tenure and therefore not contractible.\(^1\) The optimal contract will discourage myopic behavior by avoiding incentives that load excessively on short-term cash flow performance.

This requirement, together with the linear effort cost structure, implies that investors should provide exact working incentives for the agent against shirking. This binding incentive-compatibility constraint (with respect to effort deviations) is key to solving the contracting problem with private savings. In our model, to implement the interior working effort when the agent can privately save, the contract cannot specify a wage cut after the agent’s poor performance. The argument is based on the agent’s potential joint deviation of “shirking and saving.” Intuitively, a binding incentive-compatibility constraint implies that the agent loses nothing by shirking. As a result, if the contract assigns decreasing wages for no success, then because under shirking the path of no success occurs with probability one, the shirking agent who saves concurrently can strictly improve his payoff by smoothing his consumption along the path of no success.

In deriving the optimal contract, we first solve the relaxed problem which only considers the necessary conditions that rule out the agent’s local deviations, including the joint deviation described above. This

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\(^1\)This captures the cost of high-powered incentive schemes, a well-documented economic phenomenon (e.g., Levitt and Dubner (2005), Larkin (2006)). In the literature of corporate finance, this idea is connected to Stein (1989) and the ongoing literature on over-valued equity and related agency issues (e.g., Jensen (2005), and Efendi, et al. (2007)). One of the most celebrated example, citing from Larkin (2006), is Sear’s experience offering commissions to its auto mechanics based on total charges for parts and labor. Mechanics responded to this scheme by ordering unneeded repairs, and Sears ended up settling a class-action lawsuit over excessive billing.
relaxed problem allows for a recursive structure with two state variables: the agent’s marginal utility, and the agent’s continuation payoff. We then show that the obtained solution to the relaxed problem is indeed the optimal contract that solves the original problem. In the optimal contract, wages are downward rigid; the agent is guaranteed with the current wage level, and works for future pay raises (promotions). Also, when a streak of poor performance leads to an endogenous termination, the agent walks away with a severance pay that supports his post-firing consumption at the current wage level. The severance pay is increasing in his past performance, and decreasing in his outside option. These features are widely observed in practice.

Whether private savings are possible or not makes the optimal wage contract drastically different. Figure 1 compares the optimal wage process in our model, to the one derived under the same setup except that the agent’s savings are observable. For both cases, the agent starts with the same initial state, and experiences the same cash flow performance (at $t = 0.5, 1, 1.5, \text{ and } 3.5$). When savings are observable (and therefore contractible), the agent’s wages display a “zig-zag” pattern, responding actively to not only cash flow realizations, but also no cash flow realizations (which represent poor performance in this model). In contrast, when the agent can privately save, wages are only adjusted upward, never responding to poor performance. On severance pay, the fired agent leaves the company with nothing in the case with observable savings. However, the private-saving case features a positive severance pay when the agent is dismissed due to his poor performance.\footnote{There are other papers show that positive severance payments, by soothing the agent’s fear of dismissal, might provide proper incentives for risk-taking (Berkovitch, Israel, and Spiegel (2000)), or complete information disclosure (Inderst and Mueller (2005), Eisfeldt and Rampini (2008)). In essence, these findings are along the same line as this paper: By promising a generous severance package, the contract prevents the agent from harmful deviation strategies (e.g., shirking and saving in this paper; see Section 3.2).}

Several extensions are considered in Section 6, including the renegotiation-proof contract in Section 6.4. In Section 6.5, we show that our model is robust to the non-contractibility of the myopic action loss. motivated by a multi-tasking model analyzed in Holmstrom and Milgrom (1991), we allow this non-contractible loss to be reflected by a noisy but contractible measure. When the precision of this noisy measure goes to zero, the value from an optimal complete contract converges to the one from the incomplete contract that ignores this almost informationless measure. Therefore, if there exists some fixed information acquisition cost, then the contract derived in this paper could be indeed optimal in a complete contract paradigm. Also, we employ a new method in establishing the convergence result, where we do not need to solve for the optimal complete contract per se.

This paper belongs to the burgeoning continuous-time contracting literature.\footnote{This literature builds on the vast literature on discrete-time long-term agency models (Spear and Srivastava (1987); and Pheland and Townsend (1991); etc.).} Our model takes a framework similar to DeMarzo and Sannikov (2006), who study a continuous-time version of the DeMarzo and Fishman (2007) model. Biais et al. (2007) show that the contract of DeMarzo and Sannikov (2006) arises in the limit of discrete-time model. In all three papers the agent is risk-neutral, which eliminates the saving incentives.\footnote{The follow-up studies include He (2009) who studies executive compensation by analyzing a geometric Brownian motion model, Piskorski and Tchistyi (2007) who study optimal mortgage design by considering exogenous regime switching in the...} Sannikov (2008) studies an optimal contracting problem with a risk-averse agent where savings
are observable and contractible.

The problem studied in this paper is akin to the literature on unemployment insurance, e.g., Hopenhayn and Nicolini (1997). Kocherlakota (2004) solves an optimal unemployment insurance contract where the agent’s savings are private. The discrete-time model of Kocherlakota (2004) features a single success of permanent employment, but the idea of implementing interior effort under a linear effort cost structure is similar. We study a more general setup with multiple cash flows in a continuous-time framework, allow for endogenous termination with severance pay in the employment contract, and provide rigorous justifications and proofs for the optimality of the contract. For more detailed comparison to Kocherlakota (2004), including the advantage of the continuous-time analysis, see Section 6.1.3.5.6

investors’ discount rate. On the Poisson structure, Sannikov (2007) investigates an upward-jump model (as the paper here) with both moral hazard and adverse selection (the later version of the paper uses the Brownian setup), and Biais et al. (2009) who incorporate accident risk which is modeled as downward poisson jumps. Another strand of continuous-time contracting literature starts from Holmstrom and Milgrom (1987). This framework allows for private savings, due to the absence of wealth effect. See, e.g., Fudenberg, Holmstrom, and Milgrom (1990), Williams (2006), and He (2008); the latter two characterize the optimal contract with private savings.

A recent paper by Mitchell and Zhang (2007) shows that it is never optimal to implement interior effort in the setting of Kocherlakota (2004). Similar to Kocherlakota (2004), they only consider one single success, and there is no termination. Interestingly, based on CARA preference and linear additive effort costs (rather than monetary effort costs as in Holmstrom and Milgrom (1987)), they provide a nice solution to optimal contracting with private savings and binary-effort. In their analysis, allowing for negative consumption is important. In contrast, our paper imposes the realistic constraint that the agent’s consumption (therefore the wage payment in equilibrium) has to be nonnegative.

Other related literature on agency issues with access to credit market (especially hidden savings) includes Allen (1987), Bizer and DeMarzo (1999), Cole and Kocherlakota (2001), and Bisin and Rampini (2006), etc. Fundamentally, the issue of hidden savings is that hidden information (in contrast to hidden action as effort) arises during the long-term contractual relationship. Under

Figure 1: Optimal wage policies with private savings (top panel) and observable savings (bottom panel).
Harris and Holmstrom (1982) find that the downward-rigid wage is optimal. Their mechanism is fundamentally different from ours. In their learning-based model, without moral hazard issues, the first-best wage contract features a constant wage for the risk-averse agent to fully insure his productivity shocks. If the agent can quit, then a competitive labor market implies that looking forward, the agent’s future compensation has to stay above his expected productivity at any time during the employment. In other words, the agent’s ex-post participation constraint might be binding. As a result, to match the agent’s outside option, the contract will specify a wage raise in response to sufficiently good news about the agent’s productivity.\footnote{Berk, Stanton, and Zechner (2007) embeds a capital structure decision into this framework; they highlight the importance of human capital when firms choose the optimal capital structure. Lustig, Syverson, and van Nieuwerburgh (2008) develop a calibratable model based on Harris and Holmstrom (1982) to study the relationship between managerial compensation inequality and organization capital.}

The rest of this paper is organized as follows. Section 2 describes the model. Section 3 and Section 4 solve the relaxed problem recursively. In Section 5, we verify that downward rigid wages with severance pay are optimal. Section 6 discusses the optimal contract, and considers several extensions. Section 7 concludes. All proofs are given in the Appendix.

## 2 The Model

### 2.1 Technology

Consider a continuous-time infinite-horizon principal-agent model, where the risk-neutral investors (the principal) of an infinitely-lived firm hire a risk-averse agent for business operation. For any $t > 0$, the firm generates cash flows $YdN_t$ in the interval $(t - dt, t]$, where $\{N_t\}$ is a standard Poisson process with intensity $\{a\}$, and $Y$ is a positive constant. Denote $\mathcal{F}^N = \{\mathcal{F}_t^N\}_{t \geq 0}$ as the filtration generated by $\{N_t\}$. Later on we use “cash flow,” “jump,” and “success” interchangeably. The cash flows are observable and contractible.

The agent can generate at most $K$ cash flows. Even though our results holds for any finite $K$ (we use induction analysis), later for the sake of convenience we present results for the stationary case where $K \to \infty$. When the employment ceases, i.e., the agent is fired, investors can liquidate the firm’s assets for an exogenous value $L$, which is normalized to zero. This implies that early termination is inefficient. One can easily endogenize $L$ by a costly replacement with another new agent. Both the agent and investors discount future payoffs at a constant market interest rate $r > 0$.

The agent’s unobservable effort process $\{a\}$ (a $\mathcal{F}^N$-predictable process, i.e., making effort choice before knowing whether or not a cash flow occurs) controls the intensity process. There are three effort levels, i.e., $a_t \in \{0, p, \overline{p}\}$, where $\overline{p} > p > 0$ and $\overline{p} - p$ is small. The agent’s non-pecuniary personal effort cost (or enjoys private benefit if negative) when exerting $a_t$, in terms of the agent’s utilities, is $b \left( \frac{a_t}{p} - 1 \right) dt$, where $b$ is a positive constant.\footnote{The discrete structure of the agent’s action space is immaterial; the key is the linearity of the agent’s effort cost structure, and a discrete state-space framework, Fernandes and Phelan (2000) and Doepke and Townsend (2006) propose a recursive method to handle this issue in certain class of problems.} We call the lowest effort $a_t = 0$ as shirking. By shirking, the agent enjoys a private
benefit $bdt$, but the intensity of cash flow is zero. The agent can choose the working effort $a_t = p$; in this case he obtains zero private benefit, but the firm generates cash flows with a probability $pdt$ in the interval $(t - dt, t]$.

In this model, the agent can exert the highest myopic effort $\bar{p} > p$ to increase the cash flow intensity. In the spirit of Stein (1989), this myopic action is detrimental to the liquidation value $L$, as it represents the short-term performance-enhancing strategies that hurt the firm’s long-run value. We assume that these losses borne by investors, $\Delta dt$, are non-contractible.\(^9\)

There are other ways to interpret this non-contractible loss, and in this paper we keep our interpretations general. Note that $L$ can be interpreted as the firm’s on-going value after the agent is fired, and the loss due to these myopic actions might only be uncovered after the agent’s tenure.\(^10\) This idea is also similar to the multi-tasking problem studied in Holmstrom and Milgrom (1991) (see related analysis in Section 6.5). There, if the compensation contract imposes excessive incentives on the hard and easy-to-measure performance (cash flow occurrence in this model), the agent will ignore other dimensions of soft performances that are critical to the firm—for instance, refusing to collaborate with his colleagues thereby lowering their efficiency. The bottom line is that the myopic action captures the cost of high-powered incentive schemes, a well-documented fact in both economic and finance literatures (e.g., Stein (1989), Jensen (2005), and Larkin (2006), etc.).

Throughout the paper we consider the case that it is optimal to implement the working effort $a_t = p$ always. We verify the optimality of this policy later in Section 5.3.

2.2 The Agent

Utility function. The agent’s instantaneous utility from consumption is $u(c_t)$, where $u' > 0, u'' \leq 0$, and $c_t \geq 0$ is the consumption rate. When the agent is hired in the firm, his total utility $\bar{u}(c_t, a_t)$ takes an additive form, i.e.,

$$\bar{u}(c_t, a_t) = u(c_t) + b \left(1 - \frac{a_t}{p}\right). \tag{1}$$

When the agent is fired, his instantaneous utility is simply $u(c_t)$ without the effort-dependent term.

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\(^9\) As argued in their footnote 4 in Hart and Moore (1998), if investors value these liquidated assets more than the market does, then the liquidation value can be non-verifiable, therefore non-contractible. We can also formally model this idea in the following way. Assume that the liquidation value $L$ is positive and random; and whenever the agent exerts $a < \bar{p}$, the expected (discounted) liquidation value $L$ drops by at least $\Delta dt$. During liquidation, investors (as banks with specialty in locating the second-best users) handle the liquidation process, and report a liquidation value $\bar{L}$ which might differ from the true liquidation value $L$. Ruling out a third party (due to the possibility of collusion, etc.), the information revealed by the report $\bar{L}$ becomes as if non-contractible.

\(^10\) For instance, in August 2007, Dell restated down its past four years’ earnings by up to $150 million, and the executives who were responsible to this scandal have left the company. (Data source: “Dell to Restate Earnings After Probe,” http://biz.yahoo.com/ap/070816/dell_restatement.html.) Also, the agent’s higher personal cost due to myopic actions could incorporate the future extra costs (such as career concerns) borne by himself, as long as these costs are exogenous to the current employment contract.
Remark 1  The agent’s post-ring utility \( u(c_t) \) is below his total utility \( u(c_t) + b \) if he is shirking inside the firm. This specification implies that, the shirking benefit—which can be interpreted as enjoying perks or even personal satisfaction—is only available when the agent is hired in the firm. This constitutes one main difference from Kocherlakota (2004), where the unemployed agent does not have any additional benefit by simply staying in the unemployment insurance program. Therefore in contrast to Kocherlakota (2004), in our model termination/firing is invoked along the equilibrium path as a punishment mechanism.

For the working effort to be optimal, we have to rule out “extreme” wealth effects. Formally, we assume that there exists a strictly positive number \( L \) such that
\[
\inf_{c \geq 0} u'(c) = \gamma_L > 0. \tag{2}
\]
Intuitively, from the agent’s view, the monetary equivalence (marginally) of the effort cost is \( b/u' \). Therefore, condition (2) places an upper bound on the agent’s monetary cost of effort.\(^{11}\)

Though our results hold for general utility functions (see Section 6.3), in the main analysis we focus on a special form of \( u(\cdot) \), which is the modified CARA (Constant Absolute Risk Aversion) utility defined as follows:
\[
u(c) = \begin{cases} 
1 - \frac{\gamma}{\gamma_L} & \text{if } c < \frac{1}{\gamma} \ln \frac{L}{\gamma_L} \\
1 - e^{-\gamma/c} & \text{otherwise.} 
\end{cases} \tag{3}
\]
In words, to respect condition (2), we simply replace the upper part (when \( c \geq \frac{1}{\gamma} \ln \frac{L}{\gamma_L} \)) of the CARA utility by a linear function with slope \( \gamma_L > 0 \) (so the agent becomes risk-neutral when he is sufficiently wealthy). The CARA form possesses the convenient feature that the marginal utility is linear in the utility level, which simplifies our analysis. Note that different from Holmstrom and Milgrom (1987) who adopt a setting with monetary effort costs, the wealth effect is present in our model.

Private savings. In this paper the agent can privately save for the consumption smoothing purposes. As first noted by Rogerson (1985), when the agent’s utility is additive as in (1), the optimal contract without private savings features an “inverse-martingale property.” Under this property, the agent’s marginal utility follows a submartingale (i.e., the expected marginal utility increases over time). This is against the “consumption-smoothing property” if the agent can privately save.

We rule out the agent’s borrowing from a third party. Borrowing technology where a bank expects repayments from the agent is inconsistent with the agents’ private-saving technology. Our main results go through if the borrowing rate exhibits a sufficiently large spread over the saving rate \( r \), or if the agent faces a fixed borrowing limit.\(^{12}\)

\(^{11}\)Given a finite number \( K \) of cash flow jumps, the marginal utility level \( \gamma_L \) may never be reached in equilibrium.

\(^{12}\)With a fixed borrowing limit, in the optimal contract investors can max out the borrowing limit, and the agent is always borrowing constrained. The critical issue that a borrowing technology brings on is the agent’s option of default. Without complication of default, the framework with CARA preference (with monetary effort cost) with borrowing and negative consumption allows for a tractable solution with private savings (see Williams (2006), and He (2008)). With default, the key assumption is whether banks
2.3 Employment Contract

An employment contract specifies a wage process \( \{c_t \geq 0 : 0 \leq t < \tau \} \), and a lump-sum transfer \( F_\tau \geq 0 \) at the termination event \( \tau \) when the agent is fired, where \( \tau \) is the endogenous termination time. We denote such a contract as \( \Pi = \{c, F_\tau, \tau\} \), and each element is \( \mathcal{F}^N \)-adapted (performance-based compensation contract). Here, because of the agent’s limited liability, any contractual payment to the agent must be nonnegative.

The agent has zero initial wealth. For simplicity, after the agent remains unemployed forever (so his outside option is zero). The case of positive outside option is considered later in Section 6.2.

Denote the agent’s savings account balance as \( S_t \) (recall the borrowing constraint), which earns interest at the rate \( r \). Given the contract \( \Pi \), the agent’s problem is:

\[
\max_{\{a, \hat{c}, \hat{c}_\tau\}} \mathbb{E}^a \left[ \int_0^\tau e^{-rt} \left[ u(\hat{c}_t) - \frac{b}{p} (a_t - p) \right] dt + e^{-r\tau} \frac{u(\hat{c}_\tau)}{r} \right]
\]

s.t. \( dS_t = rS_t dt + c_t dt - \hat{c}_t dt \) with \( S_0 = 0, S_t \geq 0 \) for \( 0 \leq t < \tau \),

\( \hat{c}_\tau = r (F_\tau + S_\tau) \),

where \( \mathbb{E}^a [\cdot] \) indicates that the probability measure is induced by the agent’s effort policy \( \{a\} \); and \( \{\hat{c}\} \) and \( \hat{c}_\tau \) are privately observed consumptions during and after the employment, respectively. The concavity of \( u \) implies a constant consumption level \( \hat{c}_\tau \) in the agent’s post-firing life, and \( e^{-r\tau} \frac{u(\hat{c}_\tau)}{r} \) captures the (total) discounted utility after the termination.

We focus on the contract that implement working \( \{a_t = p\} \) always. The following lemma is a standard result in dynamic contracting (see, e.g., Cole and Kocherlakota (2001))

**Lemma 1** Without loss of generality, we only consider contracts that induce no savings.

Intuitively, whenever the agent wants to save, investors can do the savings for him. Therefore, we call the contract \( \Pi \) incentive-compatible and no-savings, if \( \{a\} = \{p\}, \{\hat{c}\} = \{c\}, \hat{c}_\tau = rF_\tau \) solves the problem in (4). The optimal contract solves the investors’ problem:

\[
\max_{\Pi \text{ is incentive-compatible and no-savings}} \mathbb{E} \left[ \int_0^\tau e^{-rt} (Y dN_t - c_t) dt - e^{-r\tau} F_\tau \right],
\]

where \( \mathbb{E} [\cdot] \) is under the probability measure induced by \( \{a_t = p\} \), i.e., the agent is working all the time before termination. Because the agent always enjoys some non-negative rents, in this problem the agent’s participation constraint never binds. Denote the solution to this problem as \( \Pi^* \).

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13Heuristically, the sequence of events during \((t - dt, t]\) is: 1) the agent makes effort decision \( a_t \); 2) the cash flow realization (or not) is observed; 3) the agent receives wage \( c_t \) according to his performance; and 4) the agent makes consumption/saving decision by choosing consumption \( \hat{c}_t \). This sequence ensures that the effort process is \( \mathcal{F}^N \)-predictable (i.e., does not depend on the cash flow realization at \((t - dt, t]\)), while the wage/consumption process is \( \mathcal{F}^N \)-adapted (i.e., can depend on the cash flow realization at \((t - dt, t]\)).
3 State Variables in the Relaxed Problem

We employ a relaxation method in this paper. We first analyze two state variables that help us solve the relaxed problem: 1) the agent’s continuation payoff, and 2) the agent’s marginal utility from consumption. Based on the agent’s (local) joint deviation strategy, in Section 3.3 we specify the necessary conditions on the evolutions of two state variables, and formulate the relaxed problem recursively only with these necessary conditions. Later we solve the relaxed problem in Section 4, and further verify that the obtained solution is indeed the solution to the original problem (5) in Section 5.

Our analysis is based on the stochastic calculus in jump processes (e.g., Protter (1990)), in which the following notation is required. For any $\mathcal{F}^N$-adapted right-continuous-left-limit process $\{A\}$, define its left-hand limit as:

$$A_{t-} \equiv \lim_{s \uparrow t} A_s,$$

which is $\mathcal{F}^N$-predictable. Essentially, $A_{t-} (A_t)$ is the process $\{A\}$’s time-$t$ value before (after) observing whether or not there is a cash flow realization during the interval $(t - dt, t]$.

3.1 Continuation Payoff and Incentive-Compatibility Constraint

In this paper, we use the incentive-compatibility constraint exclusively for the agent’s effort choice. In other words, we say that at any time $t$ the contract is incentive-compatible, if the agent’s single effort deviation (from working $a_t = p$, to shirking $a_t = 0$ or myopic action $a_t = \overline{p}$), while fixing the follow-up effort-consumption policies, cannot improve the agent’s value.

Given a contract $\Pi = \{\tau, \{c\}, F_\tau\}$, we introduce the agent’s continuation payoff, $W_t$, as,

$$W_t \equiv \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} u(c_s) \, ds + e^{-r(\tau-t)} \frac{u(r F_\tau)}{r} \right], \quad (6)$$

which is the agent’s future value from the contract $\Pi$ if he keeps working (so $a_t = p$ always) until termination, and conducts no savings. It is important to note that, in equilibrium, $W_t$ has to be the agent’s optimal value among all possible deviation strategies.

Using Eq. (6), the martingale representation result allows us to write the evolution of $W$ as (see the proof of Proposition 1),

$$dW_t = rW_{t-} \, dt - u(c_{t-}) \, dt + \beta^W_t (dN_t - pdt), \quad (7)$$

where $\{\beta^W_t\}$ is some $\mathcal{F}^N$-predictable process. Economically, the martingale loading $\beta^W_t$ measures the responsiveness of the agent’s continuation payoff to the unexpected performance $dN_t - pdt$ under the equilibrium working effort.

In Eq. (7), $\beta^W_t$ controls the agent’s incentives to exert effort, fixing the agent’s equilibrium consumption plans as recommended by the contract. Intuitively, the agent’s local effort decision is as follows. Choosing

\footnote{Throughout, for processes involving jumps, $dA_t$ is defined as $A_t - A_{t-} \, dt$ due to the feature of right-continuous-left-limit (RCCL) property of the standard Poisson process.}
a_t affects the agent’s effort cost b \left(1 - \frac{a_t}{p}\right) dt; however, this also sets the drift of dN_t to be a_t in his continuation payoff. As a result, the agent is solving

$$\max_{a_t \in [0,p]} b \left(1 - \frac{a_t}{p}\right) + \beta_t^W a_t.$$ 

Because the objective is linear in a_t, \beta_t^W has to equal \frac{b}{p} in order to implement the interior working effort a_t = p.

Under the framework of binary effort levels (e.g., DeMarzo and Sannikov (2006)), to motivate working against shirking, the incentive \beta_t^W must be no less than \frac{b}{p}. Because the same argument can be applied to the effort choice between “working” and “myopic action,” to prevent a = \overline{a}, \beta_t^W must be no greater than \frac{b}{p}. In words, because highly powered incentives can induce some myopic actions from the agent, investors never impose excessive incentives on the agent. As a result, \beta_t^W = \frac{b}{p}.$^{15}$ We have the following proposition, in line with Sannikov (2008):

**Proposition 1** For any employment contract \Pi to be incentive-compatible, the agent’s continuation payoff W_t evolves according to (7), and \beta_t^W = \frac{b}{p} for all t \in [0, \tau) a.e.. This implies that the agent is indifferent between working and shirking, i.e., he obtains the same value by taking a_t = 0 or p for any t \in [0, \tau).

For illustration, consider the following discrete-time example (which we will use it again in the next subsection). Ignore discounting (r = 0), and set p = 0.5, b = 2. Suppose that at date t before consumption, the agent is promised with a continuation payoff of 11. Consider a contract where the agent’s date t consumption c_t = 1, and assume that u(1) = 1. Then his post-consumption continuation payoff at t is 10. In equilibrium, for promise keeping we must have

$$0.5 \times W_{t+1}^1 + 0.5 \times W_{t+1}^0 = 10,$$

where W_{t+1}^1 (or W_{t+1}^0) is the pre-consumption continuation payoff at date t + 1 with (or without) success along the equilibrium path. This condition (8) reflects the drift in Eq. (7): with r = 0, there is no drift, therefore W_t is a martingale. Now it is clear that the reward difference W_{t+1}^1 - W_{t+1}^0 pins down the agent’s working incentives. To implement interior working, however, it must be the case that W_{t+1}^1 = 12 and W_{t+1}^0 = 8. If not, say W_{t+1}^1 = 13 (or 11) and W_{t+1}^0 = 7 (or 9), then the agent will take the myopic (or shirking) action.$^{16}$ Here, the incentive loading \beta_t^W = W_{t+1}^1 - W_{t+1}^0 = \frac{b}{p} = 4. Note that if the agent shirks, his deviation payoff is \beta_t^0 = 2 + 9 = 11 > 10 = 0.5W_{t+1}^1 + 0.5W_{t+1}^0, which is the value from working.

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$^{15}$The standard binary-effort setting without private saving issues (e.g., DeMarzo and Sannikov (2006), He (2009)) features binding incentive-compatibility constraint in the optimal contract. In contrast, here the binding incentive-compatibility constraint is directly due to the presence of myopic actions.

$^{16}$For instance, if W_{t+1}^1 = 11 and W_{t+1}^0 = 9, by shirking the agent’s deviation value is b + W_{t+1}^0 = 2 + 9 = 11 > 10 = 0.5W_{t+1}^1 + 0.5W_{t+1}^0, which is the value from working.
3.2 Marginal Utility

Now we investigate the agent’s saving incentives. Denote the agent’s marginal utility at time $t$ as $M_t \equiv u'(c_t)$. We have the following proposition, based on the requirement that the agent’s continuation payoff $W_t$ has to be the optimal value among all possible deviation strategies.

**Proposition 2** The the necessary conditions for $\Pi$ to be incentive-compatible and no-savings are:

1. The continuation payoff process $\{W\}$ evolves according to (7), where $\beta_t^W = \frac{b}{p}$ for all $t \in [0, \tau)$ a.e.

2. For all $0 \leq t < t' < \tau$, the agent’s marginal utility process $\{M\}$ satisfies $E^a_t [M_{t'}] \leq M_t$ a.e., where the agent’s action $a_s = 0$ or $p$ for $s \in (t, t')$.

To gain some intuition, we discuss the implication of Condition 2 on the equilibrium dynamics of $M$; in Section 3.3 we will state the result formally. To rule out private savings, the agent’s expected marginal utility must be non-increasing over time; otherwise, the agent can smooth his consumption and obtain a deviation value strictly higher than his equilibrium continuation payoff $W$. Heuristically, the marginal utility must satisfy:

$$E^a_t [M_t] \leq M_t.$$  

(9)

where $E^a_t [M_t]$ is the conditional expectation of $M_t$ given effort choice $a_t$ before knowing whether or not there is a cash flow in $(t - dt, t]$.

As a salient feature of any dynamic agency problem, the probability measure is induced by the agent’s endogenous effort choice $a_t$. Necessarily, under the equilibrium working effort, condition (9) requires that

$$E^{a_\tau=0}_t [M_t] = (1 - pdt) \cdot M^0_t + pM^1_t dt \leq M_t,$$

(10)

where $M^0_t (M^1_t)$ is the agent’s marginal utility at $t$ without (with) success during the interval $(t - dt, t]$.

More importantly, because the agent obtains the same payoff from shirking (recall Proposition 1), the same result must hold for the off-equilibrium shirking effort $a = 0$. Specifically, when the agent shirks—so for sure there is no jump—condition (9) requires that

$$E^{a_\tau=0}_t [M_t] = M^0_t \leq M_t.$$  

(11)

This immediately implies a surprising result that cutting wage after failure violates condition (11), because on the path of no success the agent’s marginal utility (wage) cannot rise (fall).

Intuitively, this result is derived from the agent’s potential joint deviation of shirking and saving; for a similar argument, see Kocherlakota (2004). Following the previous discrete-time example discussed in the end of Section 3.1, let us assume that $u'(1) = 1$, $u'(0.8) = 1.1$, and $u'(1.2) = 0.9$. Recall that $c_t = 1$. Consider a hypothetical wage-cut contract which assigns a lower wage after poor performance, e.g., set $c^0_{t+1} = 0.8$ and $c^1_{t+1} = 1.2$. Recall that $p = 0.5$ in this example; this wage-cut contract satisfies the
no-savings condition (10) under the measure induced by working, but violates the no-savings condition (11) under the measure induced by shirking. Here, the agent’s profitable joint-deviation under this hypothetical wage-cut contract is as follows. In the end of Section 3.1, we have shown that by deviating from working to shirking, the agent’s pre-consumption continuation payoff at $t$ is still $11$. Now the agent can concurrently save 0.1 for date $t+1$; and because there will be no success for sure tomorrow, his pre-consumption deviation value at $t$ becomes (simply assume that he follows the equilibrium strategies from $t+1$ on):

$$11 + 2u(0.9) - u(1) - u(0.8) > 11,$$

where 11 is the agent’s equilibrium payoff. Therefore this wage-cut contract fails to be incentive-compatible and no-savings.

**Remark 2** Condition (11) only states that wage cannot fall after a failure. Even combining with condition (10), they do not necessary imply downward rigid wages, because we have not yet ruled out the possibility of cutting wages after success, i.e., $M^1_t > M^-_t$ (in terms of wages, $c^1_t < c^-_t$). In Kocherlakota (2004), the downward-rigidity is easily shown because there is only one success (so the contracting problem essentially ends after a success). That simple argument does not apply in this paper with multiple cash flows.

Nevertheless, using the dynamic programming approach, later in Section 4.4.3 we rule out the case of cutting wages after success, and show that the optimal contract features

$$M^1_t \leq M^0_t = M^-_t.$$ 

Translating this condition to a statement of wages, the agent’s wages are downward-rigid: Wages remains constant without jumps, but might rise in response to successes.

**Remark 3** The linearity of effort cost and the presence of myopic action play important role in the analysis. Essentially, they force the incentive-compatibility constraint to be binding when the agent chooses working against shirking (Proposition 1). Without them, the contract may impose highly-powered incentives so that the agent finds it strictly worse off when he deviates to shirking. For instance, if there is no myopic action, then the contract can set $\beta^W_t > \frac{b}{p}$. Then by shirking, the agent loses $\beta^W_t - \frac{b}{p} > 0$, and the local shirking-saving strategy illustrated by the previous numerical example might not be profitable. As a result, we will no longer have the key condition (11). Similar argument applies to the case of convex effort cost. See related discussions in Section 5.3.2 and Section 6.5. In Conclusion we further discusses other robustness issues, including the assumption that the probability of success under shirking is 0.

### 3.3 Formulating the Relaxed Problem

We have derived the necessary conditions in Proposition 2 for the contract to be incentive-compatible and no-savings. The relaxed problem just replaces the original constraints in Problem (5) with these necessary conditions. Also, we rule out randomization in solving the relaxed problem. Because later on we show that the investors’ value function without randomization is concave, this treatment is without loss of generality.
3.3.1 Dynamics of state variables.

To be in line with the analysis of jump processes (e.g., Protter (1990), Biais et al. (2009)), we use the left-hand limit of \( W \) and \( M \), i.e., \( W_{t-} \equiv \lim_{s \uparrow t} W_s \) and \( M_{t-} \equiv \lim_{s \uparrow t} M_s \), to be the state variables.\(^{17}\)

According to Proposition 2, the (left-limit of) agent’s continuation payoff \( W_{t-} \) follows:

\[
\begin{align*}
    dW_t &= rW_{t-} dt - u(c_{t-}) dt + \frac{b}{p} (dN_t - pdt). \\
\end{align*}
\]

The agent’s marginal utility serves as the second state variable in this model.\(^{18}\) We have the following lemma which gives a formal statement of the dynamics of \( M_{t-} \). Here, \( dM^D_t \leq 0 \) in (14) corresponds to the condition (11), and \( dM^D_t \leq -\beta^M_t pdt \) in (15) corresponds to the condition (10).

**Lemma 2** Condition 2 in Proposition 2 holds if and only if there exist two \( \mathcal{F}^N \)-predictable processes \( \{\beta^M_t\} \) and \( \{M^D_t\} \) for \( t \in [0, \tau) \) such that\(^{19}\)

\[
\begin{align*}
    dM_t &= dM^D_t + \beta^M_t dN_t, \tag{13} \\
\end{align*}
\]

where

\[
\begin{align*}
    dM^D_t &\leq 0 \tag{14} \\
\end{align*}
\]

and

\[
\begin{align*}
    dM^D_t &\leq -\beta^M_t pdt. \tag{15} \\
\end{align*}
\]

3.3.2 Recursive formulation.

Recall that in our model the agent can generate at most \( K \) cash flows. Let \( W_{0-} \equiv W_0 \) and \( M_{0-} \equiv M_0 \). Denote by \( J^K(W_{0-}, M_{0-}) \) the investors’ value given the initial state variables \( W_{0-} \) and \( M_{0-} \), where \( K \) denote the number of remaining cash flows. The relaxed problem, in its recursive formulation, is

\[
\begin{align*}
    J^K(W_{0-}, M_{0-}) = \max \mathbb{E} \left[ \int_0^\tau e^{-r t} (Y dN_t - c_t) dt - e^{-r \tau} F_\tau \right], \\
\end{align*}
\]

subject to the constraints (12), (13), (14) and (15).

\(^{17}\) \( J(W_t, M_t) \) and \( J(W_{t-}, M_{t-}) \) only differ at (countably many a.e.) points where a cash flow occurs, which is a zero measure set.

\(^{18}\) At the first sight it seems that we can equivalently choose the non-decreasing wage \( c \) as another state variable. However, for potential Pareto improvement, the optimal contract should allow for a simple randomization technique, where the marginal utility becomes the key variable in preventing the agent’s private savings. We will formally show the concavity of the investors’ value function (with arguments \( W \) and \( M \)) in Proposition 4 in Section 4.4, which implies that randomization is suboptimal.

\(^{19}\) We focus on the employment path \( t \in [0, \tau) \). After the agent is fired at \( \tau \), there are no further cash flows, and consumption smoothing implies that \( dM_t = dM^D_t = 0 \).
4 Solution to the Relaxed Problem

Based on the dynamic programming technique, we solve the relaxed problem in this section in a heuristic way; in Section 5 we will formally verify that the solution indeed solves the relaxed problem. In Section 5 we also verify that the solution in fact solves the original problem; therefore it is the optimal contract that we are after.

4.1 Preliminaries

We can construct the investors’ value function $J^K(W_t, M_t)$ iteratively (see Appendix A.6). Because the key properties of value function is independent of $K$, for illustration purpose in the main text we take $K$ to infinity, and denote $J(W_t, M_t) = J^\infty(W_t, M_t)$. To save notation, most of time we will use $(W, M)$ as arguments for the value function $J(\cdot, \cdot)$, and it is understood that $(W, M)$ corresponds to $(W_t, M_t)$.

Several functions are useful in later analysis. It is clear that the agent’s marginal utility $M = \gamma_L$. Since when $M = \gamma_L$, the analysis is trivial (the agent becomes risk-neutral; see Section 4.5 for this case), we focus on the strictly concave part of (3). To express the agent’s utility and consumption in terms of $M$, we define the utility function,

$$U(M) = u(c) = 1 - \frac{M}{\gamma},$$  

and the consumption function,

$$c(M) = \frac{1}{\gamma} \ln \frac{\gamma}{M}.$$  

The fact that $U(M)$ is linear in $M$, a unique property of CARA utility, simplifies the following derivation. When the agent is fired, to fulfill the continuation payoff $W$ investors will simply pay the agent $F_r = \frac{u^{-1}(rW)}{r}$. Therefore we define the investors’ value function at termination as,

$$J^L(W) = \frac{u^{-1}(rW)}{r}.$$  

4.2 Optimal Contract and Time Line

To solve the relaxed problem, we take the guess-and-verify approach. The first step is to guess the optimal policy, which is illustrated in Figure 2. It depicts the time-line where the subperiod of $n^{th}$ cash flow is highlighted. As shown, we decompose each subperiod into the wage-setting stage and the production stage. Given an occurrence of cash flow, in the wage-setting stage investors have the option to raise the agent’s wage from $c(M)$ to $c(M^*)$, which corresponds to the optimal marginal utility response $\beta_t^M$. Then we enter the production stage, where the agent keeps working ($a_t = p$) until the $n^{th}$ cash flow realizes, or is fired before the $n^{th}$ cash flow realization.

As shown in Figure 3, $J(W, M)$, which is the value function in the wage setting stage, incorporates the investors’ option value to raise the agent’s wage before they ask the agent to work. Denote by $\tilde{J}(W, M)$ the value function in the production stage which excludes this option value.
Figure 2: Time-line for the optimal contracting. The $n^{th}$ cash flow subperiod starts with the occurrence of $(n-1)^{th}$ cash flow. Investors can raise the agent’s wage (wage-setting stage with value function $J(W, M)$) from $c(M)$ to $c(M')$. Afterwards the agent works to produce the $n^{th}$ cash flow (production stage with value function $J(W, M)$). The project is liquidated and the agent is fired if his continuation payoff $W$ hits $\frac{U(M')}{r}$ before the occurrence of the $n^{th}$ cash flow. These two stages repeat themselves for the following subperiods.

4.3 Production Stage: Construction of $\tilde{J}(W, M)$

The construction is backward. Specifically, we take as given the value function $J(W, M)$ in the wage-setting stage after the realization of the $n^{th}$ cash flow; we then move backward to consider the production and wage-setting stages in the $n^{th}$ cash flow subperiod, which is before the realization of the $n^{th}$ cash flow.

As shown in Figure 2, suppose that at time $t_{n-1}$ the $(n-1)^{th}$ cash flow occurs and both parties enter the wage-setting stage. Suppose that in the wage-setting stage investors set a marginal utility $M_{t_{n-1}}$ for the agent who has a continuation payoff $W_{t_{n-1}}$. This gives the initial state of the production stage.

4.3.1 Dynamics of state variables.

Due to (12), without success $W$ evolves as:

$$dW_t = rW_{t-}dt - U\left(M_{t-}\right)dt - bdt.$$  (19)

For the dynamics of $M$ in (13), we will verify in the next subsection that in the production stage it is optimal to set

$$dM_t^D = 0.$$  (20)

As a result, the agent’s marginal utility $M_{t-} = M$ remains constant without success. Once a cash flow occurs, the agent’s continuation payoff jumps to $W_{t-} + \frac{b}{p}$, and investors obtain a value $J\left(W_{t-} + \frac{b}{p}, M\right)$.

\footnote{Even though it seems that there is no corresponding jump $\beta_t^M$ in $M$ here, keep in mind that $J(\cdot, \cdot)$, as the value function in the wage-setting stage, has taken into account the option of reducing $M$ in response to a cash flow. In Section 4.4 we will study the optimal response of $\beta_t^M$ given a cash flow in the wage-setting stage.}
4.3.2 Termination.

In our model, one form of inefficiency comes from early termination/ring. Define the termination line
\[ l(M) = \frac{U(M)}{r} \] as shown in Figure 3. The following lemma characterizes the termination/ring.

**Lemma 3** When \( W = l(M) = \frac{U(M)}{r} \), the agent is fired and the firm is liquidated.

This result can be understood as follows. Due to potential shirking benefit, \( W - \frac{U(M)}{r} \) reflects the positive rent enjoyed by the agent. When \( W = \frac{U(M)}{r} \), zero future rent triggers an immediate firing. Here, even though the firing is due to the agent’s poor performance, he is granted a severance pay \( \frac{c(M)}{r} \) in the “punishing” termination event. Therefore we have (recall Eq. (18)):

\[
J\left(\frac{U(M)}{r}, M\right) = J^L\left(\frac{U(M)}{r}\right) = -\frac{c(M)}{r}.
\]

4.3.3 Value function \( \bar{J}(W, M) \) and its properties.

In the region \( W > \frac{U(M)}{r} \), Eq. (12) and constant \( M \) before any success imply a Hamilton-Jacobi-Bellman (HJB) equation for investors’ value function \( \bar{J} \):

\[
r \bar{J}(W, M) = pY - c(M) + p \left[ J\left(W + \frac{b}{p}, M\right) - \bar{J}(W, M)\right] + \bar{J}_W(W, M) (rW - U(M) - b) .
\]

The left-hand-side is the investors’ required return. On the right-hand-side, the first term is the expected cash flow, and the second term is the wage payment. The last two terms capture the value change due to the evolution of state variable \( W_t \): the third term is the expected value change due to the jump from \( W_t^- \) to \( W_t^- + \frac{b}{p} \), and the fourth term is the value change due to the drift of \( W_t^- \) without jump.

We will show that \( W < \frac{b+U(M)}{r} \) along the equilibrium path in the next subsection. Intuitively, \( \frac{b+U(M)}{r} \) is the upper bound of the agent’s continuation payoff \( W \) given \( M \).\(^{21}\) From another angle, \( W < \frac{b+U(M)}{r} \) implies that \( W \) decreases without success in (19), which represents punishment given failures.

Given \( W < \frac{b+U(M)}{r} \), the Ordinary Differential Equation (ODE) in (21) admits a closed-form solution:

\[
\bar{J}(W, M) = [b - rw + U(M)]^{1+\frac{1}{p}} \left[ \int_{\frac{U(M)}{r}}^W \frac{pY + pJ(x + \frac{b}{p}, M) - c(M)}{[b - rx + U(M)]^{2+\frac{1}{p}}} dx + J^L\left(\frac{U(M)}{r}\right) b^{-1-\frac{1}{p}} \right] .
\]

One can read the solution as follows: at any state \((W', M)\), investors’ instantaneous gain is simply

\[
p\left(Y + J\left(W' + \frac{b}{p}, M\right)\right) - c(M) ,
\]

which is the expected value upon success, minus the wage payment. Therefore, the investors’ value at state \((W, M)\) is the integration over these instantaneous success gains for \( W' < W \), plus the liquidation

\(^{21}\)The value \( \frac{b+U(M)}{r} \) is the agent’s guaranteed annuity utility from consumption, plus his permanent shirking benefit—which is the highest value that investors can possibly deliver given the wage level \( c(M) \).
value $J^L \left( \frac{U(M)}{r} \right)$ in the scenario that the agent is fired before he delivers any cash flow; both are properly weighted based on the Poisson structure.

We list the main properties of the production stage value function $	ilde{J}$ in Proposition 3. As the fixed-point argument suggests, they are based on the properties of $J$ in the wage-setting stage, which we will study in the next subsection.

**Proposition 3** For the production-stage, the value function $\tilde{J}(W, M)$ satisfies:

1. $\tilde{J}_W \geq -\frac{1}{\gamma L}$, and $\frac{1}{\gamma r M} < \tilde{J}_M - \frac{1}{\gamma r} \tilde{J}_W \leq \frac{1}{\gamma r \gamma L}$.

2. $\tilde{J}_{WW} < 0$, $\tilde{J}_{MM} < 0$, and $\tilde{J}_{WW} \tilde{J}_{MM} - \left( \tilde{J}_{WM} \right)^2 > 0$. Therefore $\tilde{J}(W, M)$ is concave.

3. $\tilde{J}_{WM} < 0$, and $\tilde{J}_M \left( \frac{b+U(M)}{r}, M \right) < 0$.

For property 1, because it costs investors at most $\frac{1}{\gamma L}$ to deliver one unit of $W$, $\tilde{J}_W$ is bounded by $-\frac{1}{\gamma L}$. And, as shown in Figure 3, the endogenous termination probability is determined by $w = W - \frac{U(M)}{r} = W - \frac{1}{r} + \frac{M}{\gamma r}$. Therefore, $-\tilde{J}_M + \frac{1}{\gamma r} \tilde{J}_W$ measures the (negative) impact on investors’ value function by reducing $M$ (raising the agent’s wage) while fixing the termination probability (keeping $w$ at constant), and the second estimation result follows from the fact that the pay raise has to be permanent.\(^{22}\)

The concavity of $\tilde{J}$ in property 2 implies that any randomization beyond cash flow shocks is suboptimal. To ensure concavity, we need the following sufficient condition on the project’s profitability (which is used

\(^{22}\)To keep $w$ constant, a unit decrease in $M$ has to be accompanied with $\frac{1}{\gamma r}$ units increase in $W$; this explains $-\tilde{J}_M + \frac{1}{\gamma r} \tilde{J}_W$. Also, because the future marginal utility $M_s \leq M_t = M$ where $s > t$, the marginal cost brought on by permanently reducing one unit of $M$ is a weighted average of $-\frac{\gamma (M_t)}{\gamma r}$ in the future, which must belong to $\left( \frac{1}{\gamma r M}, \frac{1}{\gamma r \gamma L} \right)$.
in the proof of Lemma 5 in Appendix A.6):

\[ Y > \max \left( \frac{1}{\gamma^r} \left[ \frac{\gamma}{\gamma_L} - 1 \right]^2, \frac{b}{P\gamma_L} \right). \]  

(23)

The third property pertains to the optimal wage-setting policy, which we will turn to in the next subsection.

4.4 Wage-Setting Stage: Construction of \( J(W, M) \)

4.4.1 Option to Raise Wage and Properties of \( J(W, M) \).

Recall that at time \( t_{n-1} \) the \((n-1)\)th cash flow occurs. Suppose that \( M_{t_{n-1}} = M \), and the agent now has a continuation payoff \( W_{t_{n-1}} = W_{t_{n-1}} + \frac{b}{P} \). If investors decide to keep the same marginal utility (i.e., set \( M_{t_{n-1}} = M = M_{t_{n-1}} \)) and enter the production stage, then we know that investors obtains a value \( \tilde{J}(W, M) \) as shown in the previous section. However, investors have the option to raise the agent’s wage (or, reduce \( M \)) and enter the production stage with a new state \((W_{t_{n-1}}, M')\). Of course, this option is valuable if investors can choose \( M' < M \) so that \( \tilde{J}(W, M') > \tilde{J}(W, M) \).

Following this idea, we define the optimal marginal utility level \( M^* \), as a function of \( W \), as

\[ M^*(W) \equiv \arg \max_{M' \in [\gamma_L, \gamma]} \tilde{J}(W, M'). \]  

(24)

We then define the investors’ value function at the wage-setting stage to be:

\[ J(W, M) \equiv \begin{cases} 
\tilde{J}(W, M') & \text{if } M \leq M^*(W) \\
\tilde{J}(W, M^*(W)) & \text{otherwise} 
\end{cases}. \]  

(25)

Simply put, whenever the realization of cash flow brings the state \((W, M)\) to be above the curve \( M^*(W) \), investors reduce \( M \) to \( M^*(W) \) by exercising the option of raising the agent’s wage, as shown in Figure 3. This implies that the optimal response of marginal utility \( M \)—i.e., \( \beta^M_t \) in (13)—to a cash realization at \( t \) is:

\[ \beta^M_t = \min \left( M^* \left( W_t - \frac{b}{P} \right) - M_{t-}, 0 \right). \]  

(26)

The transformation in (25) not only implies \( J_M \geq 0 \) always, but also allows \( J \) to inherit the concavity from \( \tilde{J} \). The following proposition gives properties of the value function \( J(W, M) \) based on Proposition 3.

**Proposition 4** For the wage-setting stage, the value function \( J(W, M) \) satisfies:

1. \( J_W \geq -\frac{1}{\gamma_L} \), and \( \frac{1}{\gamma_L} < J_M - \frac{1}{\gamma^r} J_W \leq \frac{1}{\gamma^r \gamma_L} \).

2. \( J_{WW} < 0, J_{MM} \leq 0, \) and \( J_{WW} J_{MM} - (J_{WM})^2 \geq 0 \). Therefore \( J(W, M) \) is concave.

3. \( J_{WM} \leq 0; J_M \geq 0 \) and \( J_M \left( \frac{b + U(M)}{P}, M \right) = 0 \).
4.4.2 Tradeoff of wage-setting.

The economic rationale behind the wage-setting policy is the trade-off between the termination cost and the consumption smoothing benefit. From the cost side, as \( w = W - \frac{U(M)}{r} \) captures the distance to liquidation (see Figure 3), a smaller \( M \) reduces \( w \), leading to a higher termination probability. Intuitively, given a promised continuation payoff \( W \), the agent’s future rent (beyond his wage guarantee) will be smaller for a higher wage guarantee. This implies a more stringent punishment scheme, which makes the costly termination more likely. On the benefit side, due to the agent’s risk-aversion, raising wage gives a consumption-smoothing benefit (as the agent’s equilibrium consumption pattern is back-loaded). Consequently, the optimal wage-setting policy equates the marginal cost (brought on by inefficient terminations) with the marginal benefit (for consumption smoothing).

This trade-off is reflected in property 3 in Proposition 3. First, \( \tilde{J}_{WM} < 0 \) simply says that the marginal benefit of raising wage is greater when the continuation payoff \( W \) is higher. To see this, \( \tilde{J}_{WM} < 0 \) implies that for \( W > W' \), \( -\tilde{J}_M (W, M) > -\tilde{J}_M (W', M) \), where \( -\tilde{J}_M \) captures the marginal benefit from raising wage today. In words, a higher continuation payoff \( W \) mandates investors to pay more in the future, leading to a higher consumption-smoothing benefit.

Second, \( \tilde{J}_M \left( \frac{b+U(M)}{r}, M \right) < 0 \) implies that the curve \( M^* (W) \) stays below the line \( W = \frac{b+U(M)}{r} \); put differently, it is always optimal to set a higher wage on this line. As mentioned in Section 4.3.3, \( \frac{b+U(M)}{r} \) is the upper bound of the agent’s continuation payoff \( W \) given \( M \). It implies that when \( W = \frac{b+U(M)}{r} \), once a cash flow occurs, the wage level has to increase. As a result, the marginal benefit of consumption smoothing is strictly positive. From the cost side, one can check that, starting from \( \frac{b+U(M)}{r} \), the marginal impact of future termination cost by setting \( M^* (W) \) slightly below \( \frac{b+U(M)}{r} \) is zero (see Appendix A.6.2). As a result, the benefit consideration dominates, and it is strictly optimal to raise wage when \( W = \frac{b+U(M)}{r} \). Consistent with this intuition, when the agent becomes risk-neutral at \( M = \gamma_L \), the consumption-smoothing benefit turns zero, we have \( W^* (\gamma_L) = \frac{b+U(\gamma_L)}{r} \).

Third, the deduction of \( M^* (W) \) in (24) implies that \( \tilde{J}_M (W, M^* (W)) = 0 \) for \( M^* (W) \in (\gamma_L, \gamma) \). The wage-setting curve \( M^* (W) \) is downward sloping as shown in Figure 3, because we have

\[
M''(W) = -\frac{\tilde{J}_{WM}}{\tilde{J}_{MM}} < 0.
\]

Define the inverse function \( W^* (M) \), which is the highest continuation payoff given \( M \) such that \( \tilde{J}_M \) remains nonnegative.

\( ^{23} \)To see this, according to the property 1 in Proposition 3, when \( M = \gamma_L \) we have \( \tilde{J}_M = \frac{1}{\gamma_L} \left( \tilde{J}_W + \frac{1}{\gamma_L} \right) \) always. Therefore when \( W = \frac{U(\gamma_L)+b}{r} \), the first-best result holds, and \( \tilde{J}_W = -\frac{1}{\gamma_L} \) implies \( \tilde{J}_M = 0. \)
4.4.3 Raising wages without success?

In this section we now rule out the case of raising wages without success. In other words, investors will not exercise the option of raising wages along the path without success (i.e., maintain $M$ as constant). This result has two important implications. First, this is the implicit assumption $dM_t^D = 0$ in (20) that we used in deriving $J$ in Section 4.3. Second, combining $dM_t^D = 0$ with the condition $-\beta_t^M pdt \geq dM_t^I$ in (15) gives $\beta_t^M < 0$, which further implies that cutting wages after success $M_t^I > M_t^D$ is impossible. Therefore, we rule out the concern raised in Remark 2, and in the optimal contract wages are downward-rigid.

To see this result, note that for states below $M^* (W)$, due to the construction in (25), we have $J_{WM} = \tilde{J}_{WM} < 0$. After wage-setting, the production stage must start from some state $(W_{t_{n-1}}, M)$ on or below $M^* (W)$, such that $J_M (W_{t_{n-1}}, M) \geq 0$ (see Figure 3). Suppose that the $n^{th}$ cash flow occurs at $t_n > t_{n-1}$. Then along the path without success, we have $W_t < W_{t_{n-1}}$ for $(t \in t_{n-1}, t_n]$. But $J_{WM} < 0$ implies that $J_M (W_t, M) > J_M (W_{t_{n-1}}, M) \geq 0$, and as a result raising wages (reducing $M$) is suboptimal. Intuitively, the marginal benefit of raising wages is smaller for subsequent lower continuation payoffs without success. If it is optimal to maintain the wage when $W = W_{t_{n-1}}$ initially, then it must be the case as well along the path without success.

4.5 Upper-First-Best Region

The above analysis does not cover the upper-first-best region $\{(W, M) : W \geq \frac{b+U(\gamma_L)}{r}, M = \gamma_L\}$ where the agent becomes risk neutral; see Figure 3. In that region, the optimal contract is straightforward: the risk-neutral agent with $W \geq \frac{b+U(\gamma_L)}{r}$ consumes wages never below $c (\gamma_L)$, keeps working always, and obtains $\frac{b}{p\gamma_L}$ from each cash flow realization $Y$. There is no future inefficient termination, and the first-best result is achieved. For derivations of $J$ in this upper-first-best region, see Appendix A.6.3.

5 Verification of the Optimal Contract

5.1 Verifying the Optimal Solution to the Relaxed Problem

We first verify the contract described in Section 4 solves the relaxed problem formulated in Section 3.3. For any contract II that satisfies the necessary conditions stated in Proposition 2, we introduce the investors’ auxiliary gain process $G_t$ (II) as

$$
G_t (II) = - \int_0^t e^{-rs} c_s ds + \int_0^t e^{-rs} Y dN_s + e^{-rt} J (W_t, M_t).
$$

Recall the dynamics of two state variables in (12), (13), (14), and (15):

$$
\begin{align*}
    dW_t &= rW_t dt - u (c_t-) dt + \frac{b}{p} (dN_t - pdt), \\
    dM_t &= dM_t^D + \beta_t^M dN_t, \text{ where } dM_t^D \leq 0 \text{ and } dM_t^D \leq -\beta_t^M pdt.
\end{align*}
$$
Note that here the relevant controls are $dM_t^P$ and $\beta_t^M$, and the heuristic result in Section 4 suggests that the optimal control is $dM_t^P = 0$ in (20) and $\beta_t^M = \min \left( \left( W_t^- + \frac{b}{p} \right) - M_{t^-}, 0 \right)$ in (26).

The investors’ expected instantaneous gain, which captures the drift of $G_t$ (scaled by $e^{rt}$), is (note that $W = W_t^-$ and $M = M_t^-$)

$$
\mathbb{E}_{t^-} \left[ e^{rt} dG_t \right] = \left[ -rJ(W_t^-, M) - c(M) + p \left( Y + \left[ J \left( W_t^-, M + \beta_t^M \right) - J(W_t^-, M) \right] \right) 
+ J_W \cdot (rW_t^- - U(M) - b) 
+ \left[ J(W_t^-, M + dM_t^P) - J(W_t^-, M) \right] \right] dt.
$$

In the proof of Proposition 5, we show that the optimal policy to maximize $\mathbb{E}_{t^-} \left[ e^{rt} dG_t \right]$ is setting $dM_t^P = 0$ and $\beta_t^M = \min \left( \left( W_t^- + \frac{b}{p} \right) - M_{t^-}, 0 \right)$, which exactly correspond to the optimal contract described in Section 4. Due to the construction in Section 4, we have $\mathbb{E}_{t^-} \left[ e^{rt} dG_t \right] = 0$ under the optimal policy, and $\mathbb{E}_{t^-} \left[ e^{rt} dG_t \right] \leq 0$ for other incentive-compatible and no-savings contracts. Then the standard verification argument leads to the following proposition, which shows that our contract in Section 4 solves the relaxed problem formulated in Section 3.3. Finally, since $J$ is concave, randomization cannot improve the investors’ value.

**Proposition 5** Consider the stationary case $K \to \infty$. The investors’ value function $J(W, M) = J^\infty(W, M)$ exists with properties established in Proposition 4, and the wage-setting curve $M^\star(W)$ defined in (24) satisfies $M'^\star(W) < 0$. Under the optimal solution to the relaxed problem formulated in Section 3.3, we have $W$ evolves according to (12) and $M$ evolves according to (13), where $dM_t^P = 0$ and $\beta_t^M = \min \left( \left( W_t^- + \frac{b}{p} \right) - M_{t^-}, 0 \right)$.

### 5.2 Verifying the Optimal Contract for the Original Problem

Now we show that the solution to the relaxed problem is the solution to the original problem. The key observation is that, under the downward-rigid wage contract in Proposition 5, the agent’s optimal strategy is to exert working effort and maintain zero savings always (for a formal argument, see the proof of Theorem 1 in Appendix A.7.1). In other words, the obtained solution not only satisfies the necessary conditions identified in Proposition 2, but also satisfies the tighter constraints (i.e., a smaller set of feasible contracts) imposed by the original problem (5). As a result, the solution to the relaxed problem is indeed the solution to the investor’s original problem (5). Therefore we have the following main theorem of the paper.

**Theorem 1** Under the optimal contract $\Pi^\star$ that implements working, we have

$$
dW_t = (rW_t^- - U(M_{t^-}) - b) dt + \frac{b}{p} dN_t,
$$

and $dM_t^P = 0$, $\beta_t^M = \min \left( \left( W_t^- + \frac{b}{p} \right) - M_{t^-}, 0 \right)$ so that

$$
dM_t = \beta_t^M dN_t.
$$
The employment is terminated whenever \( W_t = \frac{U(M_t)}{r} \), and the agent gets a severance pay \( F_t = \frac{c(M_t)}{r} \). When \( W_t > \frac{U(\gamma_L) + b}{p} \) and \( M_t = M^* (W_t - \gamma_L) = \gamma_L \), the first-best result is achieved: Investors pay the agent \( \frac{1}{\gamma_L} [W_t - \frac{U(\gamma_L) + b}{r}] \), ask him to work forever, and pay him \( \frac{b}{p \gamma_L} \) whenever a cash flow occurs.

5.3 Optimalty of Implementing Working

In this section we verify that, under certain sufficient conditions, it is optimal to implement working always. Skipping this section will not hinder the reading of Section 6.

When a certain action \( a_t \) is implemented at time \( t \), the evolution of \( W \) follows:

\[
dW_t = rW_t dt - U(M_{t-}) dt + \frac{b}{p} (a_t - p) dt + \beta_t^W (dN_t (a_t) - a_t dt),
\]

where \( \beta_t^W \leq (\geq) \frac{b}{p} \) if \( a_t = 0 (p) \) (the proof will be similar to that of Proposition 1; see also Sannikov (2008)). Our task is to ensure that \( \mathbb{E}_{t-} [e^{rt} dG_t] \leq 0 \) when investors implement actions other than working.

5.3.1 Suboptimality of shirking.

Suppose that at time \( t \) shirking is implemented; then we must have

\[
dW_t = rW_t dt - U(M_{t-}) dt + \frac{b}{p} (0 - p) dt + \beta_t^W dN_t (a = 0),
\]

where \( \beta_t^W \leq \frac{b}{p} \). Because there is no success when shirking is implemented, \( dN_t (a_t = 0) = 0 \). Moreover, to prevent the agent from saving we must have \( dM_t = dM^D_t \leq 0 \). Since \( J_M \geq 0 \), it is optimal to set \( dM^D_t = 0 \), and

\[
e^{rt} dG_t \leq \left[ -rJ - c(M) + J_W \cdot (rW - U(M) - b) \right] dt.
\]

Due to the construction of \( \tilde{J} \) (therefore \( J \)) in the ODE (21), we need the following condition to ensure \( e^{rt} dG_t \leq 0 \):\(^{25}\)

\[
Y + J \left( W + \frac{b}{p}, M \right) - J (W, M) \geq 0 \text{ for all } (W, M).
\]

Because \( J \) is concave in \( W \), \( J (W, M) - J \left( W + \frac{b}{p}, M \right) \geq -J_W (W, M) \frac{b}{p} \). Since property 1 in Proposition 4 implies that \( \frac{1}{\gamma_L} \geq -J_W \), we have the following sufficient condition:

\[
Y \geq \frac{b}{p \gamma_L}. \tag{28}
\]

\(^{24}\)Here using \( \tau \) or \( \tau^- \) makes no difference, because termination cannot occur on the point of cash flow realization (given a success, the agent’s continuation payoff \( W_t - \frac{b}{p} > \frac{U(M)}{r} \) given \( W_t \geq \frac{U(M)}{r} \)).

\(^{25}\)The ODE in (21) is about \( \tilde{J} \); but because \( J (W, M) = \tilde{J} (W, \min(M^* (W), M)) \), the same ODE holds for \( J \), which says that

\[
rJ = pY - c(M) + p \left[ J \left( W + \frac{b}{p}, M \right) - J \right] + J_W \cdot (rW - U(M) - b).
\]

Therefore we have \(-rJ - c(M) + J_W \cdot (rW - U(M) - b) = -p \left[ Y + J \left( W + \frac{b}{p}, M \right) - J \right] \).
This condition is ensured by the parameter restriction in (23). Moreover, the above condition is also necessary to rule out shirking, because it is the standard condition for the suboptimality of shirking when the agent becomes risk-neutral in the upper-first-best states (where $J$ is linear in $W$). Intuitively, for working to be optimal, the expected cash flow $pY$ should be greater than the upper-bound of the agent’s equivalent “monetary” effort cost, which is $b/\gamma_L$ when the agent becomes sufficiently wealthy.

5.3.2 Suboptimality of myopic actions.

If the myopic action $a = \bar{p}$ is implemented at $t$, then for some $\beta_t^W \geq \frac{b}{\bar{p}}$, we can write the evolution of $W_t$ as:

$$dW_t = rW_{t-}dt - U(M_{t-})dt + \frac{b}{\bar{p}}(\bar{p} - p)dt + \beta_t^W(dN_t(\bar{p}) - \bar{p}dt).$$

We need to show that the auxiliary gain process $G$ in (27) has a negative drift once $\bar{p}$ is implemented.

Let us pause to discuss the economic intuition. There is a non-contractible loss $\Delta$ due to the myopic action. On the benefit side, the myopic action boosts the cash flow intensity to $\bar{p}$. We envision that the gain $\epsilon \equiv \bar{p} - p$ is small. Are there any other gains by implementing the myopic action in this model?

The answer is yes. In Remark 3 we note that the binding incentive-compatibility constraint $\beta_t^W = \frac{b}{\bar{p}}$ plays a key role in applying the joint-deviation argument in Section 3.2. Now when $\beta_t^W > \frac{b}{\bar{p}}$, the agent’s incentive-compatibility constraint is slack, and condition (11) no longer holds. In other words, under a highly-powered incentive scheme, the optimal contract punishes shirking severely and therefore deter the agent’s joint-deviation of “shirking and saving.” As a result, cutting the agent’s wage after his failure—which is a potentially value-improving policy because of $J_M > 0$—becomes possible.

In this case, because the unidimensional variable $M$ is no longer sufficient to capture the agent’s private-saving incentives (which depend on the entire continuation contract), it is difficult to pinpoint the benefit of adjusting the wage downward. Fortunately, we can use the necessary (local) no-savings condition under the effort choice $a = \bar{p}$ to bound this benefit. We can write the evolution of $M$ as:

$$dM_t = dM_t^D + \beta_t^M dN_t(a_t = \bar{p})$$

$$= dM_t^D - \beta_t^M \bar{p}dt + \beta_t^M (dN_t(a_t = \bar{p}) - \bar{p}dt).$$

Then the no-savings condition under $a_t = \bar{p}$ requires that $dM_t$ has a non-positive drift (supermartingale):

$$dM_t^D \leq -\beta_t^M \bar{p}dt.$$

Because after a jump it must be true that $M_{t-} + \beta_t^M \geq \gamma_L$, we have a more explicit bound for $dM_t^D$ which captures wage reduction without success (note that $M_{t-} - \gamma_L > 0$):

$$dM_t^D \leq (M_{t-} - \gamma_L)\bar{p}dt,$$

(29)
Essentially, this condition places a bound on the increment \( dM^D_t \), which in turn gives a bound for the gain in cutting wages. Based on (29), in Appendix A.8.1 we derive a sufficient condition for the loss \( \Delta \) to offset the upper-bound estimate of this gain:

\[
\Delta > \max_{M \in [\gamma_L, \gamma]} p(M - \gamma_L) J_M \left( \frac{U(M)}{r}, M \right).
\]

(30)

Because the actual gain (subject to additional constraints regarding the agent’s other deviating strategies) must be lower, we provide a sufficient condition for the suboptimality of implementing the myopic action.

5.3.3 Verifying the optimality of working.

By combining the above results, we have the following proposition. The proof is based on the standard verification argument.

**Proposition 6** Under the conditions (28) and (30), it is optimal to implement working always.

6 Discussions and Extensions

6.1 Optimal Wage Contract

We now discuss the optimal wage contract. In the optimal contract, the agent is promised with a life-time wage guarantee. If the agent’s performance is sufficiently good, he will receive pay raises (as promotions), and these raises are permanent. On the other hand, given poor performance, the agent is dismissed with a severance pay to support his post-firing consumption at his current wage level, and he loses potential future pay raises.

6.1.1 Comparison to the case with observable savings.

The possibility of private savings has a dramatic impact on the optimal wage policy. Keep the same model, but consider the only modification that the agent’s savings are (publicly) observable. In this case, the agent’s consumption, which is just the wage paid by investors, is contractible. As a dynamic agency problem with hidden actions studied in Sannikov (2008), the agent’s continuation payoff \( W_t \) is the only state variable in solving for the optimal contract (see Appendix A.9), and the optimal wage becomes a function of the continuation payoff \( W_t \).

We graph the optimal wage policies (left scale) and associated continuation payoff dynamics (right scale) in Figure 4. The history consists of 4 cash flows occurred in \( t = 0.5, 1.0, 1.5, \) and 3.5; afterwards the agent generates no cash flows even with his effort input. The top (bottom) panel is for the case with observable (private) savings; they are just the examples that we used in Figure 1 in Introduction. For better comparison, we use the same scale for both cases.

\[26\] In the binary-effort version of Sannikov (2008), there are no myopic actions. But because in Sannikov (2008) the optimal contract features a binding incentive-compatibility constraint, the restriction brought on by myopic actions is redundant.

24
In the top panel with observable savings, the agent’s wages exhibit a quite sensitive response (a zig-zag pattern) to his performance: The wage goes up for any success, and drops given no success. In contrast, in the bottom panel with private savings, the response is muted: Wages are downward rigid, and pay raises are less frequent (only twice given four cash flows). Put differently, the agent’s wage might go up or stay the same following successes, and he never gets pay cut after poor performance. The incentive is only reflected in the responsiveness of the agent’s continuation payoff $W$, which corresponds to the value of his future pays or stock options in reality.

Finally, the wage policy also has noticeable impact on the termination policy. Given the long poor performance after $t = 3.5$ in Figure 4, in the top panel with observable savings, the agent’s continuation payoff falls at a lower rate than in the bottom panel; this is due to the downward wage adjustment along the path of poor performance. As a result, with observable savings (the top panel), the life span of the firm is longer (22.23) than that of the case with private savings (17.62). Besides, in contrast to zero severance pay when savings are observable, the agent in our model walks away with a positive severance pay.

The seemingly inefficient compensation patterns in the bottom panel, i.e., low compensation-performance sensitivities and generous severance payments even after poor performance, are usually viewed as symptoms of malfunctioned corporate governance (e.g., Bebchuk and Fried (2004)). However, this paper shows that under realistic contracting friction assumptions, they are actually part of the optimal contract. By providing a concrete example, this paper raises a critique to the Bebchuk and Fried’s logic flow from the observed “inefficient” forms of executive compensation to the failed corporate governance (for a similar point, see Core, Guay, and Thomas (2004)).

6.1.2 Consumption contract vs. wage contract

It is worth emphasizing that we are only characterizing the optimal consumption contract, i.e., the optimal amount that the agent should consume given his performance history. This raises the question whether our optimal wage contract is unique, because the wage (compensation) contract is only an implementation of the desired consumption contract. A closely related question is, even though in the derived contract investors save for the agent, can the agent save for himself (hence truly private savings) while still achieving the desired consumption policy? Interestingly, since we rule out two-way transfers between investors and the agent (i.e., both wage $c$ and severance pay $F$ have to be nonnegative), the implementation is indeed unique before the region of $M = \gamma_L$ is reached. To see this, when $M > \gamma_L$, along the equilibrium path it is always possible that the agent will be borrowing constrained (whenever $\beta_t^M < 0$ so resetting a higher wage is possible). Therefore at these states, the agent, if given any positive private savings, will consume strictly more than the level stipulated by the optimal contract.
Figure 4: Optimal wage policies and associated continuation payoff evolutions for the cases with private savings (the bottom panel) and the case with observable savings (the top panel). The solid line is for wage process, and the dotted line is for the agent’s continuation payoff $W$. The history consists of 4 cash flows at $t = 0.5, 1.0, 1.5, and 3.5$; and no cash flow afterwards. Parameters are $b = 0.5, Y = 20, r = 0.2, p = 0.5, \gamma = 5$, and $\gamma_L = 1$. 

Termination $\tau = 22.33$, with zero severance pay

Termination $\tau = 17.62$, with severance pay $c/r = 1.32$
6.1.3 Comparison to Kocherlakota (2004).

The main result in this paper is similar to Kocherlakota (2004) who studies a discrete-time model with single success. We use the techniques recently developed by Sannikov (2008) to analyze a continuous-time model with Poisson structure, and solves the general case with multiple successes. As emphasized in Remark 2, the main result of downward-rigidity is far from trivial in the setting of multiple successes. Moreover, we show that randomization is suboptimal, and verify the optimality of implementing working always (as opposed to implementing shirking or myopic actions sometimes); both steps are missing from Kocherlakota (2004).

In this paper, we have endogenous terminations as a form of disciplinary mechanism, which is absent from Kocherlakota (2004). As pointed out in Remark 1, the driving difference is that in Kocherlakota (2004)’s unemployment insurance setting, the agent’s utility is independent of whether he is inside the unemployment insurance program or not. In contrast, in our firm-manager setting, it is natural to assume that the agent may enjoy some strictly positive perks—either pecuniary or nonpecuniary such as personal satisfaction—only during his tenure inside the firm. Moreover, the endogenous termination/firing mechanism not only enriches our analysis, but also reflects plausible economics: In our model, when the agent is fired, he loses future option value of being promoted—which captures certain aspects of top managers’ career concerns.

Finally, a continuous-time framework employed here simplifies the analysis greatly. One important advantage is that we do not need to tackle the randomization issue around termination. Specifically, if the time is discrete, then in Figure 3 along the equilibrium path, $W$ might drop below the termination line $l(M)$. As a result, a randomization scheme (potentially 2-dimensional!) is required before $W$ hitting $l(M)$ to respect the agent’s promise-keeping constraint.

6.1.4 Comparison to Harris and Holmstrom (1982).

Besides the above discussion regarding the “inefficiency” of executive compensation, our model admits broader implications, and can be applied to any long-term labor compensation contracts where workers’ incentives are important. Harris and Holmstrom (1982) also derive downward-rigid wage contracts to be the optimal contract. In that model, learning about the agent’s ability and the firm’s one-sided commitment are the driving forces. In contrast, we obtain the same dynamic structure for the optimal contract under a framework with moral hazard only. In this regard, the theoretical predictions from our model is consistent with the empirical evidence mentioned in Harris and Holmstrom (1982), i.e., the positive relationship between experience and earnings, and the positive skewness of earnings, etc.

Even though Harris and Holmstrom (1982) and this paper generate similar results, it is possible to separate these two theories empirically. Start with an agent who just receives a pay raise, and focus on how the ordering of his follow-up performance (i.e., whether successes comes before failures) affects his next

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27This issue is similar to the difference between the discrete-time model in DeMarzo and Fishman (2007) and the continuous-time model in DeMarzo and Sannikov (2006). Biais et al (2007) show this difference in a more formal way.
pay raise. In typical learning models such as Harris and Holmstrom (1982), the agent’s performance is his underlying ability plus some i.i.d. noises, and the ordering of the agent’s performance does not matter. It is because the simple average of these performance is the sufficient statistic in updating the agent’s perceived ability, which determines the agent’s pay raise if any. In contrast, in our model earlier successes lead to a higher continuation payoff $W$ (due to the simple discounting effect $r > 0$; check (12)), and as a result the following pay raise should be greater.\(^{28}\)

6.2 Severance Package and Outside Option

The size of severance package depends on the agent’s outside option once he leaves the firm. In the base model the agent has zero outside option; now suppose that he will receive a constant $z$ in perpetuity after his layoff.\(^{29}\) As the agent has an outside option $\frac{u(z)}{r}$, the admissible continuation payoff $W$ should be above this level. Geometrically, in the state space $(W, M)$ in Figure 3, we need to impose an extra restriction that $W \geq \frac{u(z)}{r}$.

With this positive outside option, the severance package becomes $\max [c(M) - z, 0] / r$ to prevent the agent’s consumption from falling after the layoff. Therefore the severance package is decreasing with the agent’s outside option.

6.3 General Utility Functions

The adoption of CARA utility is only for exposition purposes. This section extends our analysis to a general utility function $u(\cdot)$ that satisfies the bounded-below marginal utility in condition (2). Similar to (16), by writing $g(c) = u'(c)$, we define the agent’s utility, as a function of the marginal utility $M$, to be $U(M) = u\left(g^{-1}(M)\right)$. Now the termination boundary $l(M) = \frac{U(M)}{r}$ is no longer a line as in the CARA case (see Figure 3). For the concavity of the value function $J$, we require that the domain $\{(W, M) : W \geq l(M)\}$ to remain convex—in other words, $l(M)$ is a convex function. One can easily check that $l(M)$ is convex if and only if $u'' > \left(\frac{u''}{u'}\right)^2$, a property that is also satisfied by the class of power utility.

The structure of resulting optimal contract remains unchanged: wages $\{c\}$ are nondecreasing before $M$ reaches its lower bound; the agent works for potential pay raises; and the agent’s poor performance leads to dismissal, but he walks away with a severance payment $\frac{z'r}{r}$. In Appendix A.10 interested readers can find detailed constructions for the general utility case.

\(^{28}\)Consider the following two histories after the first pay raise: $(1, 1, 0, 1)$ and $(0, 1, 1, 1)$, where 1 (0) indicates success (failure); and suppose that there is a second pay raise after these four realizations. Our model predicts a greater pay raise in the first path, while Harris and Holmstrom (1982) imply that we should observe a same pay raise for both paths. Of course the ordering could matter in learning models if the noise variance is time-varying. However, under the reasonable assumption that time-varying variances are independent with the agent’s performance, on average we should not expect significant relationship between performance ordering and compensation.

\(^{29}\)This can be interpreted as unemployment insurance program. Note that the perpetuity of payment is immaterial. Suppose that the unemployment insurance only pays out $z$ over $T$ years. Then effectively the infinitely-lived agent has an outside option of $z' = z (1 - e^{-rT})$, given his optimal consumption smoothing.
6.4 Renegotiation-Proof Contract

In this model, because termination is ex-post inefficient, both parties tend to renegotiate whenever the original contract can be Pareto improved ex post. For the contract to be renegotiation proof, the value function $J^{RP}(W, M)$ must have a non-increasing slope with respect to the agent’s continuation payoff $W$; otherwise, both party can be strictly better off by raising $W$. For simplicity, we only consider a moderate impact of renegotiation by focusing on the case where the termination inefficiency is not excessive, i.e., a relatively high liquidation value $L$ (see Appendix A.11 for details).

In Appendix A.11 we construct the value function $J^{RP}(W, M)$ recursively. Analogous to the unidimensional result in DeMarzo and Sannikov (2006), $J^{RP}(W, M)$ features a renegotiation boundary $W(M)$ with $J_{W}(W(M), M) = 0$. The renegotiation curve $W(M)$ is the lower bound of the agent’s continuation payoff $W$ along the equilibrium employment path at the wage level $c(M)$. When the liquidation value $L$ is relatively high, $W(M)$ (which might bind at $\frac{U(M)}{r}$) is strictly below the wage-setting curve $W^*(M)$; see the left panel in Figure 5.

We have similar results for the renegotiation-proof optimal contract. However, when poor performance drives $W$ down to $W(M)$, both parties run a lottery, whose outcome is independent of the cash flow occurrence. The agent is fired (so $W$ becomes $\frac{U(M)}{r}$ and he loses $W(M) - \frac{U(M)}{r}$) with a probability

$$\frac{b + U(M) - rW(M)}{W(M) - \frac{U(M)}{r}} dt;$$

The definition of renegotiation-proofness here is the same as in DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006), which is equivalent to the contract being sequentially undominated (in terms of parties payoffs); see Hart and Tirole (1988). In contrast, Hart and Moore (1998) use a different approach. See related comments in DeMarzo and Fishman (2007).
otherwise, the agent stays at \( W(M) \). Under this lottery, at \( W = W(M) \), without success the agent’s (expected) \( dW \) remains \( [rW(M) - U(M) - b] \, dt \) as in (19). The right panel in Figure 5 gives an example of \( J^{RP}(W, M) \) as a function of \( W \) (fixing \( M \)). As shown, \( J^{RP}(W, M) \leq 0 \), and \( J^{RP} \) is flat with respect to \( W \) in the region of \( \left[ \frac{U(M)}{\tau}, W(M) \right] \), reflecting the randomization (lottery) between \( \frac{U(M)}{\tau} \) and \( W(M) \). For detailed constructions and proofs, see Appendix A.11 and related results in DeMarzo and Sannikov (2006).

6.5 The Complete Contract with Multi-Tasking: A Convergence Result

As argued before, we envision the myopic actions as the situation where once equipped with excessive incentives, the agent will be motivated to hurt the firm in certain ways that investors either cannot specify the damages in some verifiable terms ex-ante, or are only able to discover future losses after the agent’s tenure. By treating the myopic action loss \( \Delta \) as non-contractible, our contracting space is incomplete. How far is our optimal contract away from the optimal complete contract? To address this issue, we embed a multi-tasking problem (a la Holmstrom and Milgrom (1991)) into the main model. Assume that the firm’s operation involves another business activity, which generates an instantaneous value increment as,

\[
dQ_t = -\Delta 1_{\{a_t = \bar{\rho}\}} dt + \sigma dZ_t,
\]

where \( \{Z_t\} \) is a Brownian process independent of \( \{N_t\} \). Here \( dQ_t \), as the agent’s second soft performance measure, is observable and contractible. We can also interpret \( dQ_t \) as the (noisy) change of the firm’s long-run value. To capture the “softness” of the \( dQ \) measure, in the following analysis we consider the case where \( \sigma \) is sufficiently large.

Neither shirking or working has any impact on the drift in \( dQ_t \). But once the agent takes the myopic action \( a = \bar{\rho} \), the drift becomes \( -\Delta \) as the agent transfers his effort allocation from the soft performance \( dQ_t \) to the hard performance \( dN_t \). Due to the risk-neutrality of investors, if the resulting complete contract chooses to ignore \( dQ_t \) completely, then it is just the contract derived in Section 4.

When the loss \( \Delta \) is contractible through \( dQ_t \), investors can raise the incentive loading on \( dN_t \) but still prevent the agent from taking myopic actions. The contract can specify an incentive scheme such as:

\[
dW_t = (rW_{t-} - U(M_{t-})) \, dt + \beta_t^{W} (dN_t - pdt) + x_t dQ_t,
\]

where the incentive loading \( \beta_t^{W} = \frac{b}{\bar{p}} + k_t > \frac{b}{\bar{p}} \) and \( k_t > 0 \). Now if we set

\[
x_t = \frac{k_t (\bar{\rho} - p)}{\Delta} = \frac{k_t \epsilon}{\Delta},
\]

then the agent will be refrained from the myopic action: By taking \( a = \bar{\rho} = p + \epsilon \), the agent gains \( k_t \epsilon \) from \( dN_t \), but this gain is offset by the loss \( x_t \Delta \) from \( dQ_t \).

As discussed in Section 5.3.2, setting \( \beta_t^{W} > \frac{b}{\bar{p}} \), i.e., \( k_t > 0 \), gives rise to a benefit in relaxing the no-savings constraint, as investors may specify \( dM_{t-} > 0 \) (cut wages) on the path without success. However, setting \( k_t > 0 \) and in turn \( x_t = \frac{k_t \epsilon}{\Delta} > 0 \) is costly. This is because by imposing positive loading \( x_t > 0 \)
on the agent’s continuation payoff in (31), the noise in \( dQ_t \) makes inefficient terminations more likely. In addition, it is also inefficient to expose the risk-averse agent to random noises. This trade-off leads to the following proposition.

**Proposition 7** When \( \sigma \to \infty \) so that \( dQ_t \) information becomes extremely “soft,” the value from the optimal complete contract converges to the one from the incomplete contract derived in Section 4.

Intuitively, when the information precision of the soft performance goes to zero, the contract should simply ignore such extremely noisy signals, just as the incomplete contract does. This implies that, however small is the positive transaction cost in procuring the extremely “soft” information \( dQ_t \), the “incomplete” contract derived in Section 4 can be optimal even in the paradigm of complete contracts.

In addition, in proving Proposition 7, we employ a new method in establishing the convergence result. There, one can give some estimate for the potential gains from incorporating new contractible variables, *without solving for the exact form of optimal complete contract*. The convergence result then can be established by studying these estimates.

### 7 Conclusion

We study a dynamic agency problem where the agent can privately save. When ruling out private savings, previous studies (Rogerson (1985), Sannikov (2008), etc.) derive a front-loaded, performance-sensitive compensation flow in the optimal contract. In contrast, the optimal wage process in this paper becomes back-loaded, and relatively insensitive to performance.

Our optimal contract features a downward-rigid wage structure, and a seemingly “generous” severance pay even after the agent’s poor performance. Both patterns, which are commonly observed in today’s compensation packages, have received wide criticism due to their “suboptimality” in providing incentives efficiently. Therefore, this paper delivers a general message that, under realistic contracting frictions—such as private savings and non-contractible myopic action studied in this paper—certain seemingly inefficient contracting features can indeed be optimal.

We solve the optimal contracting with private savings by utilizing the binding incentive-compatibility constraint in the presence of myopic actions, where the linearity of effort cost structure is important. However, in justifying the non-contractibility via information acquisition costs in Section 6.5, we employ a proof method which allows the agent’s cost structure to be convex, and show the convergence result when the convexity diminishes. Therefore, our contracting result is generic in this regard.

We emphasize that the resulting contract form—especially the strict downward-rigid wage—is specific to our particular setting. Take the base model, and suppose that the probability of success under shirking is strictly positive (say \( \varepsilon \)) instead of 0. Then the necessary condition for the evolution of marginal utility is \( (1 - \varepsilon dt) M_t^0 + \varepsilon dt M_t^1 \leq M_{t-} \), and the wage after failure \( M_t^0 \) will go up (so the wage \( c_t^0 \) goes down)
in general. However, the important lesson from our analysis is that, the downward speed depends on the off-equilibrium measure implied by shirking. As $M_t^1 > 0$, the upward drift of $M$ after failure is bounded by $\varepsilon M_t^1 - dt$, and downward drift of wage given failure vanishes as $\varepsilon$ goes to zero. Clearly, the choice of zero intensity of success under shirking in the paper helps us obtain the main theoretical result in its stark contrast to that of observable savings.

Therefore, a less responsive wage pattern and a positive severance pay, which are designed to reduce the agent’s deviation values, should be robust when the agent can privately save. Of course, the exact degree of robustness needs future theoretical work to explore more general settings, which might give further guidelines in solving the optimal contracting problems with private savings.

References

Allen, F. 1987, “Repeated Principal-Agent Relationships with Lending and Borrowing,” Economic Letters 17, 27-31;


## A Appendix

### A.1 Proof of Lemma 1

Suppose that under $\Pi = \{\{c\}, F_\tau, \tau\}$, the agent’s optimal consumption-saving strategy is $\{\hat{c}_t \neq c_t; S_t \geq 0\}$. Consider offering the contract $\hat{\Pi} = \{\{\hat{c}\}, \hat{F}_\tau = \frac{\hat{c}}{c}, \tau\}$ (instead of $\Pi$) to the agent; clearly this contract just replicates the agent’s optimal consumption profile under $\Pi$. Now we show that the agent will not deviate under the new contract $\hat{\Pi}$. Suppose not; then there is a saving path $\{\hat{S} \geq 0\}$ combined with other action profile $\{\hat{a}'\}$ to support a consumption profile $\{\hat{c}'\}$ that achieve a strictly higher value for the agent. But then $\{\hat{S}' = \hat{S} + \hat{S} \geq 0\}$ with the action profile $\{\hat{a}'\}$ can support $\{\hat{c}'\}$ under the original contract $\Pi = \{\{c\}, F_\tau, \tau\}$, contradiction to the optimality of $\{\hat{c}_t \neq c_t; S_t \geq 0\}$ under the original contract $\Pi$.

### A.2 Proof of Proposition 1

Take the zero saving policy as given. Under the preassumption that $a_t = p$ for all $t$, the agent’s value process is $V_t = E_t \left[ \int_0^t e^{-rt} u(c_t) \, dt + e^{-rT} u(c_T) \right]$, and the martingale representation theorem (e.g., Biais et al. (2009)) implies that there exists an $\mathcal{F}_N$ predictable process $\{\beta_s^W\}$ such that

$$V_t = V_0 + \int_0^t e^{-rs} \beta_s^W (-pdS + dN_s).$$

According to the definition of $W_t$, we have

$$V_t = \int_0^t e^{-rs} u(c_s) \, ds + e^{-rt} W_t;$$

then differentiate both sides we obtain the expression in (7).

Now consider any feasible effort process $a = \{a_t \in \{0, p, \bar{p}\} : t \in [0, \tau]\}$. The agent’s associated value process $V_t (a)$ could be written as

$$V_t (a) = V_0 + \int_0^t e^{-rs} \beta_s^W (-pdS + dN_s (a_s)) + \int_0^t e^{-rt} \frac{b}{p} (p - a_s) \, ds$$

where $dN_s (a_s)$ has an intensity of $a_s$. Then

$$dV_t (a) = e^{-rt} \beta_s^W (-pdS + dN_t (a_t)) - e^{-rt} \frac{b}{p} (a_t - p) \, dt$$

$$= e^{-rt} (a_t - p) \left( \beta_s^W - \frac{b}{p} \right) \, dt + e^{-rt} \beta_s^W (dN_t (a_t) - a_t dt) .$$

Therefore, to implement working it must be the case that $(a_t - p) \left( \beta_s^W - \frac{b}{p} \right) \leq 0$ for both $a_t = 0$ and $a_t = p$. This implies that $\beta_s^W = \frac{b}{p}$, a binding incentive-compatibility constraint. It directly follows that the agent obtains the same value by taking any action process $\{a\}$ s.t. $a_t = 0$ or $p$. Q.E.D.
A.3 Proof of Proposition 2

This first result is just Proposition 1. Now we prove the second result. Note that in the following proof we allow for randomization other than the agent’s Poisson performance in the contract. Suppose not. Then the contract must specify some paths on \([0, T]\) with strictly positive measure so that \(\mathbb{E}_t^a[M_{t'}] > M_t\) for some action process \(a\) and \(t' > t\). Collect these time points into a set \(T\) with positive Lebesgue measure (in time), so that on this set \(T\) (indexed by the element \(t \in T\)) the marginal utility follows a submartingale (in expectation it is increasing).

Now consider the following profitable consumption smoothing strategy on this set \(T\), in which the agent saves a bit in the beginning of \(T\) and consumes in the end of \(T\). Pick the lowest (highest) \(t\)’s to form \(T_l(T_h) \subset T\) so that the Lebesgue measure of \(T_l\) is \(\varepsilon > 0\), where \(T_l(T_h)\) has a higher (lower) marginal utility. Choose \(\varepsilon\) to be sufficiently small so that

\[
T_l \equiv \inf T_l < t_l^2 \equiv \sup T_l < t_h^1 \equiv \inf T_h < t_h^2 \equiv \sup T_h,
\]

and without loss of generality we set \(t_l^1 = \inf T_l = \inf T = 0\). At \(t_l^2\) the agent’s marginal utility (wage) is strictly lower (higher) than that at \(t_h^1\) so that

\[
\mathbb{E}_t^a\left[M_{t^2}\right] < \mathbb{E}_t^a\left(M_{t^1}\right).
\] (32)

Otherwise, \(\mathbb{E}_t^a\left(M_{t^2}\right) = \mathbb{E}_t^a\left(M_{t^1}\right)\) for any small \(\varepsilon > 0\), plus the fact that \(M_t\) is a submartingale, immediately imply that \(M_t\) is martingale on set \(T\) a.e., contradiction to the construction of \(T\).

Suppose now the agent saves \(e^{-t}\) for \(t \in T_l\) and consumes \(e^{t'}\) for \(t' \in T_h\); clearly this satisfies the savings technology (if at \(T_l\) in some states wages are zero, then only consider saving on the states with strictly positive wages, and consume these savings at \(T_h\)). The total utility loss from lowering consumption on \(T_l\) is

\[
\mathbb{E}_t^a\left[\int_{T_l} e^{-rt} \left( u(c_t) - u(c_t - e^{t'} e) \right) dt \right] = \mathbb{E}_t^a\left[\int_{T_l} \left[ M_t e + o(\varepsilon) \right] dt < \varepsilon \mathbb{E}_t^a\left(M_{t^2}\right) + o(\varepsilon),
\]

because \(\mathbb{E}_t^a\left(M_t\right) < \mathbb{E}_t^a\left(M_{t^2}\right)\) on \(t \in T_l\). Similarly, the utility gain from raising consumption on \(T_h\) is

\[
\mathbb{E}_t^a\left[\int_{T_h} e^{-rt} \left( u(c_t) - u(c_t - e^{t'} e) \right) dt \right] = \mathbb{E}_t^a\left[\int_{T_h} \left[ M_t e + o(\varepsilon) \right] dt > \varepsilon \mathbb{E}_t^a\left(M_{t^1}\right) + o(\varepsilon).
\]

Therefore the total gain

\[
\mathbb{E}_t^a\left[\int_{T_l} e^{-rt} \left( u(c_t) - u(c_t - e^{t'} e) \right) dt \right] - \mathbb{E}_t^a\left[\int_{T_h} e^{-rt} \left( u(c_t) - u(c_t - e^{t'} e) \right) dt \right] = \varepsilon\left[\mathbb{E}_t^a\left(M_{t^1}\right) - \mathbb{E}_t^a\left(M_{t^2}\right)\right] + o(\varepsilon).
\]

When \(\varepsilon\) is sufficiently small, this is dominated by the first term, which is strictly positive due to (32).

A.4 Proof of Lemma 2

“If” part is obvious. Now we prove the “only if” part. Let us take the equilibrium effort process \(\{a = p\}\) first. Then according to the Doob-Meyer decomposition theorem (see, e.g., Karatzas and Shreve (1988)) and the martingale representation theorem (see, e.g., Biais et al. (2009)), there exist an \(\mathcal{F}^N\)-predictable process \(\{\beta_t^M\}\) and a predictable non-increasing process \(\{H_t^D\}\) such that

\[
dM_t = dH_t + \beta_t^M (dN_t - pdt),
\]

with \(dH_t \leq 0\). Define \(dM_t^D \equiv dH_t - \beta_t^M pdt\) so that \(M_t^D\) is also a predictable process. Then

\[
dM_t = dM_t^D + \beta_t^M dN_t,
\]

and since \(dH_t \leq 0\) we have \(dM_t^D \leq -\beta_t^M pdt\). We need to further prove that \(dM_t^D \leq 0\). Suppose not; say that \(dM_t^D > 0\) holds for some paths with strictly positive measure. Then if the agent takes the effort \(a = 0\) on these path, \(dN_t = 0\), and \(dM_t\) is strictly increasing on these paths with strictly positive measure. This contradicts with Condition 2 in Proposition 2.
A.5 Proof of Lemma 3

Clearly firing the agent delivers the continuation payoff of \( W = \frac{U(M)}{r} \). Now we show that there are no other ways to deliver \( W = \frac{U(M)}{r} \). We have two steps to go, and in the following argument can be understood as \( t^- \) as the information at \((t - dt, t]\) is irrelevant.

1) Note that to respect condition (9), given a marginal utility \( M \), any continuation payoff \( W < \frac{U(M)}{r} \) is infeasible. The argument is as follows. In light of Proposition 2, for any equilibrium effort policy \( \alpha \), no savings implies that \( M_t \geq \mathbb{E}_t^\alpha (M_s) \) for \( s > t \) (\( s \) could be larger than \( \tau \), in which case the agent is fired and the distribution is degenerate). According to the definition of \( W_t \) which is the agent’s optimal value, we have

\[
W_t \geq \mathbb{E}_t^\alpha \left[ \int_t^\infty e^{-r(s-t)} U(M_s) \, dt \right] \geq \int_t^\infty e^{-r(s-t)} U(M_s) \, dt \geq \frac{U(M_t)}{r},
\]

where the first “\( \geq \)” is due to the possibility of \( M_s = \gamma_L \), the second “\( \geq \)” is due to the convexity of \( U(\cdot) \) (in the CARA case it is a linear function; see related discussions in Section 6.3); and the third “\( \geq \)” is because \( U(\cdot) \) is decreasing.

2) The necessary condition (12) implies that at the point \( W = \frac{U(M)}{r} \), \( W \) is a martingale. Since \( W \) cannot fall, it has to be remain constant \( \frac{U(M)}{r} \) from then on. Because the agent obtains the same payoff by shirking and working, this implies zero potential shirking benefit. Therefore in this case the agent is fired.

A.6 Appendix for Section 4

Given \( K \) which is total number of potential cash flows, we use \( i \leq K \) to indicate the number of cash flows remaining, and we are solving for \( j^{i-1,K} \). But in our setting, since only the number of cash flows remaining matters, \( J^{i,K} \) is independent of \( K \). Therefore we omit \( K \) in the following analysis.

A.6.1 Production Stage

When \( i = 0 \), there is no future cash flows, and the firm is obsolete. Based on the definition of \( J^L(W) \) in (18), we have

\[
J^0(W, M) = \begin{cases} 
J^L(W) & \text{if } W \geq \frac{U(M)}{r} \\
-\infty & \text{otherwise} 
\end{cases}
\]

It is clear that \( J^0(W, M) \) satisfies all conditions in Proposition 4. Now consider \( i \geq 1 \). The next lemma translates Proposition 4 to the corresponding properties of \( j^{i-1,31} \).

Lemma 4 For the wage-setting stage value function \( j^{i-1} \), we have the following properties:

1. \( j^{i-1}_{w} \geq -\frac{1}{\gamma_L} \), and \( \frac{1}{\gamma_L} < j^{i-1}_m \leq \frac{1}{\gamma_L} \).
2. \( j^{i-1}_{ww} < 0, j^{i-1}_{wm} < 0, j^{i-1}_m > 0 \) and \( j^{i-1}_{ww} j^{i-1}_{mm} - \left( j^{i-1}_{wm} \right)^2 \geq 0 \). Therefore \( j^{i-1}(w, m) \) is concave.
3. \( \frac{1}{\gamma_L} j^{i-1}_{ww} + j^{i-1}_{wm} < 0, \frac{1}{\gamma_L} j^{i-1}_m + j^{i-1}_m \geq 0 \), and \( \frac{1}{\gamma_L} j^{i-1}_w \left( \frac{b}{r}, m \right) + j^{i-1}_m \left( \frac{b}{r}, m \right) = 0 \).

We carry out our analysis based on the following linear transformation,

\[
\begin{cases} 
w = W - \frac{U(M)}{r} \in \left[ 0, \frac{b}{r} \right], \\
m = M \in [\gamma_L, \gamma],
\end{cases}
\]

31Strictly speaking, here all the second-order derivatives—\( j_{ww}, j_{wm}, \) and \( j_{mm} \)—are in the weak sense (in a Soboslov space) which allows for (finite) discontinuities, and the integration-by-parts formula still holds. To be precise, in the production stage \( j^i \) is a mollified version of \( j^{i-1} \), which makes everything smooth; but the wage-setting stage only keeps the first-order smoothness (hence for the 2nd-order derivatives there will be a discontinuity on \( M^* (W) \)). However, because the first-order derivatives are continuous, the negative definiteness of Hessian matrix is sufficient for the concavity.
where the domain is a rectangle. Let \( \tilde{J}^i(w, m) = \tilde{J}^i(W, M) \), and \( j^i(w, m) = J^i(W, M) \). Clearly \( \tilde{j} \) is concave if and only if \( \tilde{J} \) is concave. Note that

\[
\tilde{J}^i = j^i, \quad \tilde{J}^i = \frac{1}{\gamma^r} \tilde{j}^i + \tilde{j}_m^i, \quad \text{and} \quad \tilde{J}^i_{WM} = \frac{1}{\gamma^r} \tilde{j}^i_{ww} + \tilde{j}^i_{wm},
\]

and similar relations hold between \( j \) and \( J \).

Without jump, \( \tilde{j}^i \) satisfies the following ODE

\[
(r + p) \tilde{j}^i(w, m) = -c(m) + p \left( Y + j^{i-1} \left( w + \frac{b}{p}, m \right) \right) + j_w(w, m) (rw - b),
\]

and its closed-form solution is

\[
\tilde{j}^i(w, m) = \frac{r}{r + p} J^L \left( \frac{U(m)}{r} \right) + \frac{p}{r + p} \left[ b - rw \right]^{1 + \frac{p}{r}} \left[ \int_0^w \frac{(r + p) \left( Y + j^{i-1} \left( x + \frac{b}{p}, m \right) \right)}{\left[ b - rx \right]^{2 + \frac{p}{r}}} dx + \frac{J^L \left( \frac{U(m)}{r} \right)}{b^{1 + \frac{p}{r}}} \right],
\]

where we use \( c(m) = -rJ^L \left( \frac{U(m)}{r} \right) \). The solution in (22) in the main text is identical to (34).

Based on (33) and (34), a direct calculation (where we use integration-by-parts formula) yields

\[
\frac{dJ^L \left( \frac{U(m)}{r} \right)}{dm} = \frac{1}{\gamma^r m}
\]

Notice that

\[
\frac{r}{r + p} + \frac{p}{r + p} \left[ b - rw \right]^{1 + \frac{p}{r}} \left[ \int_0^w (r + p) \left[ b - rx \right]^{-2 - \frac{p}{r}} dx + b^{-1 - \frac{p}{r}} \right] = 1
\]

which constitutes a probability measure. Since \( j_m^{i-1} \in \left[ \frac{1}{\gamma^r m}, \frac{1}{\gamma^r m} \right] \), we have \( \tilde{j}_m^i \in \left[ \frac{1}{\gamma^r m}, \frac{1}{\gamma^r m} \right] \).

Based on (33) and (34), a direct calculation (where we use integration-by-parts formula) yields

\[
\tilde{j}_w = \frac{p}{b - rw} \left( Y + j^{i-1} \left( w + \frac{b}{p}, m \right) \right)
\]

\[
= -p \left[ b - rw \right]^{\frac{p}{r}} \left[ \int_0^w (r + p) \left( Y + j^{i-1} \left( x + \frac{b}{p}, m \right) \right) \left[ b - rx \right]^{-2 - \frac{p}{r}} dx + J^L \left( \frac{U(m)}{r} \right) b^{-1 - \frac{p}{r}} \right]
\]

\[
= p \left[ b - rw \right]^{\frac{p}{r}} \left\{ \left[ Y + j^{i-1} \left( x + \frac{b}{p}, m \right) \right] \left[ b - rx \right]^{-1 - \frac{p}{r}} \right\} \left[ \int_0^w j_w^{i-1} \left( x + \frac{b}{p}, m \right) \left[ b - rx \right]^{-1 - \frac{p}{r}} dx + J^L \left( \frac{U(m)}{r} \right) b^{-1 - \frac{p}{r}} \right]
\]

\[
\therefore \tilde{j}_w = p \left[ b - rw \right]^{\frac{p}{r}} \left\{ \int_0^w j_w^{i-1} \left[ b - rx \right]^{-1 - \frac{p}{r}} dx + \left[ Y + j^{i-1} \left( \frac{b}{p}, m \right) - J^L \left( \frac{U(m)}{r} \right) \right] b^{-1 - \frac{p}{r}} \right\}
\]

Therefore

\[
\tilde{j}_w > \left[ b - rw \right]^{\frac{p}{r}} \left\{ \int_0^w p j_w^{i-1} \left[ b - rx \right]^{-1 - \frac{p}{r}} dx + j_w^{i-1} \left( \frac{b}{p}, m \right) b^{-1 - \frac{p}{r}} + py b^{-1 - \frac{p}{r}} \right\}
\]

37
The second inequality follows from the following fact: note that \( J^L \left( \frac{U(m)}{r} \right) = j^{i-1} (0, m) \), and \( j^{i-1} \) is concave, which implies that \( j^{i-1} \left( \frac{b}{p}, m \right) - J^L \left( \frac{U(m)}{r} \right) = j^{i-1} \left( \frac{b}{p}, m \right) - \frac{b}{p} \). Since

\[
\int_0^w p \left[ b - rx \right]^{-1 - \frac{\beta}{2}} dx + b^{-\frac{\beta}{2}} = 1
\]

which constitutes a probability measure, from (36) we know that \( \tilde{j}_w > j^{i-1}_w \geq -\frac{1}{\gamma L} \). Also, in the limiting case \( w = \frac{b}{p} \), we have

\[
\tilde{j}_w \left( \frac{b}{r}, m \right) = j^{i-1} \left( \frac{b}{r} + \frac{b}{p}, m \right)
\]

simply because when \( w \to \frac{b}{p} \), the entire probability weights in (37) are put on \( w = \frac{b}{p} \).

Now we study the second-order derivatives. It is straightforward that

\[
-\tilde{j}_{ww}^i = \frac{r}{r + p} \frac{1}{\gamma \tau m^2} + \frac{p}{r + p} \left[ b - rw \right]^{1 - \frac{\beta}{2}} \left[ \int_0^w \frac{p \left( j^{i-1}_w \left( x + \frac{b}{p}, m \right) \right)}{\left[ b - rx \right]^{1 + \frac{\beta}{2}}} dx + \frac{1}{\gamma \tau m^2} \left( b - w \right)^{-\frac{\beta}{2}} \right]
\]

This shows that \( \tilde{j}^i \) is concave in \( m \). For \( \tilde{j}_{ww} \), we use (33) and (36), and find that

\[
-\tilde{j}_{ww}^i = \frac{p}{b - rw} \left[ \tilde{j}_{ww} - j^{i-1}_w \left( w + \frac{b}{p}, m \right) \right] > \left[ b - rw \right]^{-1 - \frac{\beta}{2}} \left( b - \frac{b}{p} \right)^2 \left( b - w \right) \left[ b - rw \right]^{-1 - \frac{\beta}{2}} \left( w + \frac{b}{p}, m \right)
\]

Invoking the integration-by-parts technique again, we have

\[
\int_0^w p j^{i-1}_w \left[ b - rx \right]^{-1 - \frac{\beta}{2}} dx + j^{i-1}_w \left( \frac{b}{p}, m \right) \left( b - w \right) > j^{i-1}_w \left( w + \frac{b}{p}, m \right)
\]

and therefore

\[
-\tilde{j}_{ww}^i > \left[ b - rw \right]^{-1 - \frac{\beta}{2}} \left( b - \frac{b}{p} \right)^2 \left( b - w \right) + p \left[ b - rw \right]^{1 - \frac{\beta}{2}} \int_0^w \left( -j^{i-1}_w \right) \left[ b - rx \right]^{-\frac{\beta}{2}} dx > 0.
\]

Shortly we will need a stronger estimate for the global concavity of \( \tilde{j} \). According to (23), \( Y > \frac{1}{\gamma \tau} \left( \frac{j^{i-1}_w}{\gamma L} - 1 \right)^2 \), and

\[
-\tilde{j}_{ww} > \left[ b - rw \right]^{-1 - \frac{\beta}{2}} \left[ \int_0^w p \left( -j^{i-1}_w \right) \left[ b - rx \right]^{-\frac{\beta}{2}} dx + \frac{p^2}{b^2 \gamma \tau} \left[ \frac{j^{i-1}_w}{\gamma L} - 1 \right]^2 b^{-\frac{\beta}{2}} \right]
\]

Finally, we calculate

\[
\frac{\partial}{\partial m} \tilde{j}_{ww} = \frac{\partial}{\partial m} \tilde{j}_w = \left[ b - rw \right]^{1 - \frac{\beta}{2}} \left[ \int_0^w \frac{p j^{i-1}_w \left[ b - rx \right]^{-1 - \frac{\beta}{2}} dx + p \frac{1}{\gamma \tau m^2} \left( \frac{j^{i-1}_m}{\gamma L} - 1 \right)^2 \left( b - rw \right) - 1 \right]^{1 - \frac{\beta}{2}} b^{-\frac{\beta}{2}} \right]
\]

it immediately implies that \( \tilde{j}_{ww} \geq 0 \), because \( j^{i-1}_w \geq 0 \) and \( j^{i-1}_w \left( \frac{b}{p}, m \right) > j^{i-1}_m \left( 0, m \right) = \frac{1}{\gamma \tau m^2} \).

Now we show that \( \tilde{j}_w^i \), in fact, is globally concave, which requires that \( \tilde{j}_{ww} > \left( \tilde{j}_{ww} \right)^2 \). To show this, we invoke the Cauchy-Schwartz inequality. Observe that the terms other than the integral in \( \tilde{j}_{ww} \), and \( \tilde{j}_{ww}^i \) are \( \frac{p^2}{b^2 \gamma \tau} \left[ \frac{j^{i-1}_m}{\gamma L} - 1 \right]^2 \), \( \frac{1}{\gamma \tau m^2} \geq \frac{1}{\gamma \tau m^2} \), and \( \frac{p}{b^2} \left[ j^{i-1}_m \left( \frac{b}{p}, m \right) - \frac{1}{\gamma \tau m^2} \right] \) respectively; and we have

\[
\frac{p^2}{b^2 \gamma \tau} \left[ \frac{j^{i-1}_m}{\gamma L} - 1 \right]^2 \geq \frac{b}{b - rw} \left( \frac{b}{r + p} \right)^{1 + \frac{\beta}{2}} \left( b - w \right)^{-\frac{\beta}{2}} \left( \gamma \tau m^2 \right)^{1 + \frac{\beta}{2}} \geq \frac{1}{\gamma \tau m^2}
\]

Finally, we calculate

\[
\frac{\partial}{\partial m} \tilde{j}_{ww} = \frac{\partial}{\partial m} \tilde{j}_w = \left[ b - rw \right]^{1 - \frac{\beta}{2}} \left[ \int_0^w \frac{p j^{i-1}_w \left[ b - rx \right]^{-1 - \frac{\beta}{2}} dx + p \frac{1}{\gamma \tau m^2} \left( \frac{j^{i-1}_m}{\gamma L} - 1 \right)^2 \left( b - rw \right) - 1 \right]^{1 - \frac{\beta}{2}} b^{-\frac{\beta}{2}} \right]
\]

it immediately implies that \( \tilde{j}_{ww} \geq 0 \), because \( j^{i-1}_w \geq 0 \) and \( j^{i-1}_w \left( \frac{b}{p}, m \right) > j^{i-1}_m \left( 0, m \right) = \frac{1}{\gamma \tau m^2} \).

Now we show that \( \tilde{j}_w^i \), in fact, is globally concave, which requires that \( \tilde{j}_{ww} > \left( \tilde{j}_{ww} \right)^2 \). To show this, we invoke the Cauchy-Schwartz inequality. Observe that the terms other than the integral in \( \tilde{j}_{ww} \), and \( \tilde{j}_{ww}^i \) are \( \frac{p^2}{b^2 \gamma \tau} \left[ \frac{j^{i-1}_m}{\gamma L} - 1 \right]^2 \), \( \frac{1}{\gamma \tau m^2} \geq \frac{1}{\gamma \tau m^2} \), and \( \frac{p}{b^2} \left[ j^{i-1}_m \left( \frac{b}{p}, m \right) - \frac{1}{\gamma \tau m^2} \right] \) respectively; and we have

\[
\frac{p^2}{b^2 \gamma \tau} \left[ \frac{j^{i-1}_m}{\gamma L} - 1 \right]^2 \geq \frac{b}{b - rw} \left( \frac{b}{r + p} \right)^{1 + \frac{\beta}{2}} \left( b - w \right)^{-\frac{\beta}{2}} \left( \gamma \tau m^2 \right)^{1 + \frac{\beta}{2}} \geq \frac{1}{\gamma \tau m^2}
\]
Then, the standard Cauchy-Schwartz argument yields that

\[
\tilde{\gamma}_i \tilde{J}^i_{uw} \tilde{J}^i_{wm} > [b - rw]^{2\gamma} \int_0^w p \left( j^{-1}_{uw} j^{-1}_{wm} \right)^{\frac{1}{\gamma}} [b - rx] - \frac{1}{\gamma} \left( b - \frac{w}{\gamma} \right)^2 dx + \left( b - \frac{w}{\gamma} \right)^2 \left( b - \frac{w}{\gamma} \right)^2 \geq \left( \tilde{\gamma}_i \tilde{J}^i_{uw} \right)^2
\]

where we use the fact that \( j^{-1}_{uw} \) is concave.

Finally, we show the property 3. According to (23), \( Y > \frac{b}{p r} \). Utilizing (40) and (39), and since \( j^{-1}_{uw} \left( \frac{b}{p}, m \right) - \frac{1}{\gamma m} < \frac{1}{\gamma} \), we have

\[
\frac{1}{\gamma r} j^{-1}_{uw} \tilde{J}^i_{uw} + \tilde{J}^i_{uw} < [b - rw]^{\frac{1}{\gamma}} \int_0^w p \left( \frac{1}{\gamma r} j^{-1}_{uw} \right) [b - rx] - \frac{1}{\gamma} \left( b - \frac{w}{\gamma} \right) dx + [b - rw]^{\frac{1}{\gamma}} \int_0^w p \left( \frac{1}{\gamma r} j^{-1}_{uw} \right) [b - rx] - \frac{1}{\gamma} \left( b - \frac{w}{\gamma} \right) dx
\]

The first item is negative since \( \frac{1}{\gamma r} j^{-1}_{uw} + \tilde{J}^i_{uw} < 0 \). Because \( j^{-1}_{uw} < 0 \), and for \( x < w \) we have

\[
[b - rw]^{\frac{1}{\gamma}} [b - rx] - \frac{1}{\gamma} \left( b - \frac{w}{\gamma} \right) > [b - rw]^{\frac{1}{\gamma}} [b - rx] - \frac{1}{\gamma},
\]

the second item is negative too. Therefore \( \frac{1}{\gamma r} j^{-1}_{uw} \tilde{J}^i_{uw} + \tilde{J}^i_{uw} < 0 \).

The second inequality in property 3 says that \( \tilde{J}^i_{M} \left( \frac{U(M)+b}{r} + \frac{b}{p}, M \right) < 0 \). To show this, When \( W = \frac{U(M)+b}{r} \), take derivative w.r.t \( M \) on equation (21), one finds that

\[
r \tilde{J}^i_{M} = -c' \left( M \right) + p \left( j^{-1}_{M} \left( \frac{U(M)+b}{r} + \frac{b}{p}, M \right) - \tilde{J}^i_{M} \right) + \frac{1}{\gamma} \tilde{J}^i_{W} \Rightarrow \tilde{J}^i_{M} = \frac{1}{\gamma (r + p)} \left( \frac{1}{M} \right) + \tilde{J}^i_{W}
\]

where we use \( j^{-1}_{M} \left( \frac{U(M)+b}{r} + \frac{b}{p}, M \right) = 0 \) (Proposition 4, Property 3), and \(-c' \left( M \right) = \frac{1}{\gamma M} \). However, we have shown that \( \tilde{J}^i_{W} \left( \frac{U(M)+b}{r} + \frac{b}{p}, M \right) = j^{-1}_{W} \left( \frac{b}{p} + \frac{b}{p}, m \right) \) in (38). Now, as verified in the next wage-setting stage, investors raise the agent’s wage, and as a result there exists \( m^* < m \) so that

\[
\tilde{J}^i_{W} = \frac{b}{r} + \frac{b}{p}, m \) \left( b + \frac{U(m) - U(m^*)}{r} + \frac{b}{p}, m^* \right) < \gamma \tilde{J}^i_{M} \leq \frac{1}{\gamma m^*} \leq \frac{1}{m}
\]

Therefore \( \tilde{J}^i_{W} < -\frac{1}{\gamma M} \), and \( \tilde{J}^i_{M} \left( \frac{U(M)+b}{r}, M \right) < 0 \). Q.E.D.

**A.6.2 Wage-setting Stage**

First we show the zero marginal cost brought on by the future termination at \( W = \frac{b+U(M)}{r} \). Notice that raising wage at \( W = \frac{b+U(M)}{r} \) is equivalent to setting \( w \) below \( \frac{b}{r} \). Consider the policy of setting \( W^* \left( M \right) \) so that \( w = \frac{b}{r} - \varepsilon \). Then starting from \( W^* \left( M \right), M \), it is easy to check that the expected discounted termination probability is \( \left( \frac{\varepsilon}{\gamma} \right)^{\frac{r}{\gamma + r}} \) on the path without any jumps. Under the Poisson setup, the total expected discounted termination probability—by integrating over all jumps but with the same \( M \)—is still in the order of \( \varepsilon^{\frac{r}{\gamma + r}} \); and notice that it is an upper-bound estimator, as if a jump leads to a lower \( M^* \) then the impact on the probability of future terminations is zero. Therefore reducing \( M \) has a zero marginal impact on \( \varepsilon = 0 \) when \( p > 0 \).

Next we present a formal construction of \( J^i \) from \( \tilde{J} \). Given \( M^* \left( W \right) \) defined in the main text (note that \( M^* \) might be \( i \)-dependent), we propose a transformation

\[
T \left( W, M \right) = \left( W, \min \left( M, M^* \left( W \right) \right) \right)
\]

(41)
and define $J^i(W,M) = \tilde{J}^i(T(W,M))$. This transformation preserves the concavity. To see this, consider any two points $(W_1, M_1)$ and $(W_2, M_2)$ and

$$W(\lambda) = \lambda W_1 + (1 - \lambda) W_2$$

and $M(\lambda) = \lambda M_1 + (1 - \lambda) M_2$.

For $S = T(W(\lambda), M(\lambda))$ and $S' = \lambda T(W_1, M_1) + (1 - \lambda) T(W_2, M_2)$, both have the same $W$, but $S$ has a larger $M$. Because both $S$ and $S'$ are in the region where $J^i_M \geq 0$, we have $J^i(S) \geq J^i(S')$. Therefore

$$J^i(W(\lambda), M(\lambda)) = J^i(T(W(\lambda), M(\lambda))) \geq J^i(\lambda T(W_1, M_1) + (1 - \lambda) T(W_2, M_2)) \geq \lambda J^i(T(W_1, M_1)) + (1 - \lambda) J^i(T(W_2, M_2)) = \lambda J^i(W_1, M_1) + (1 - \lambda) J^i(W_2, M_2).$$

It is easy to check that the resulting $J^i(W,M)(j^i(w,m))$ satisfies all properties stated in Proposition 4 (Lemma 4). For completeness, we provide several properties of $j^i$ on the domain above the curve $M^*(W)$. Notice that

$$j^i(w, m) = J^i(W, M) = \tilde{J}^i(W, M^*(W)) = \tilde{J}^i\left(W - \frac{U(m^*)}{r}, m^*\right),$$

where $m^* = M^* > M$. By construction, $J^i_M(W, M) = \frac{1}{\gamma r} j^i_w(w, m) + j^i_m(w, m) = 0$. Then utilizing the fact that $J^i_M(W, M^*(W)) = 0$ (therefore the indirect impact on $m^*$ (or $M^*$) is zero), one can easily verify that

$$j^i_w(w, m) = \tilde{J}^i_w\left(W - \frac{U(m^*)}{r}, m^*\right)$$

$$j^i_m(w, m) = -\frac{1}{\gamma r} j^i_w(w, m) = \tilde{J}^i_m\left(W - \frac{U(m^*)}{r}, m^*\right) \geq \frac{1}{\gamma r m^*} \in \left(\frac{1}{\gamma r m}, \frac{1}{\gamma r r_m}\right).$$

$$\frac{1}{\gamma r} j^i_w + j^i_m = \tilde{J}^i(W, M) = 0,$$

and $j^i_{mm} = \frac{1}{\gamma r^2} j^i_{ww}$.

### A.6.3 Convergence and the Upper-First-Best States

Let $C(X)$ as the set of continuous, bounded and concave functions on the convex compact set

$$X = \left\{(W, M) : M \in [\gamma_L, \gamma], W \in \left[\frac{U(M)}{r}, \frac{U(M) + b}{r}\right]\right\} \subset \mathbb{R}^2.$$

We have defined an operator $\mathcal{O} : C(X) \rightarrow C(X)$ to construct $J^1 = \mathcal{O}\left(J^{i-1}\right)$ successively. Specifically, for $J^{i-1} \in C(X)$, define $J^i$ in two steps. First,

$$J^i(W, M) = [b - rW + U(M)]^\frac{1}{2} \left[\frac{W}{U(M)} \frac{pY - c(M) + pJ^{i-1}\left(x + \frac{b}{r}; M\right)}{b - rX + U(M)^{1-2-\frac{b}{r}}} dx + J^L\left(\frac{U(M)}{b} M\right)^{\frac{b}{r}} \frac{b^1-\frac{b}{r}}{r}\right],$$

where $J^{i-1}\left(\frac{U(M)}{b}; M\right) = J^L\left(\frac{U(M)}{b}\right)$. Second, the transformation $T(W, M)$ defined in (41) gives

$$J^i(W, M) = \tilde{J}^i(T(W, M)).$$

Now we show that mapping $\mathcal{O}$ satisfies Blackwell’s sufficient conditions for a contraction mapping (Stokey and Lucas (1989)), which implies that there exists a unique $J$ such that $J^i$ converges to $J \in C(X)$ uniformly.

We need to verify the monotonicity condition,

$$\mathcal{O}(f) \leq \mathcal{O}(g) \text{ if } f \leq g, f, g \in C(X).$$

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and the discounting condition
\[ \mathbb{Q}(f + x) \leq \mathbb{Q}f + \frac{p}{r + p}x \quad \text{where } f \in \mathcal{C}(X), \ x \in \mathbb{R} \]
To see the monotonocity condition, decompose \( \mathbb{Q} \) into \( \mathbb{Q}_1 \) (from \( J^{i-1} \) to \( J^i \)) and \( \mathbb{Q}_2 \) (from \( J^i \) to \( J^f \)). If \( f \leq g \), then \( \mathbb{Q}_1 f \leq \mathbb{Q}_1 g \).

Fix \( W \), and let \( M^*_f \) and \( M^*_g \) be the corresponding wage-setting curves. Clearly if \( M < \min(\{M^*_f, M^*_g\}) \) then \( \mathbb{Q}_2 f \leq \mathbb{Q}_2 g \) holds. If \( M > \max(\{M^*_f, M^*_g\}) \),
\[ \mathbb{Q}_2 (f)(W, M) = \mathbb{Q}_1 (f)(W, M^*_f) \leq \mathbb{Q}_1 (g)(W, M^*_f) \leq \mathbb{Q}_2 (g)(W, M^*_g) = \mathbb{Q}_2 (g)(W, M) \]
Finally consider \( M \) sits between \( M^*_f \) and \( M^*_g \). W.l.o.g. consider \( M^*_f < M^*_g \). Then
\[ \mathbb{Q}_2 (f)(W, M) = \mathbb{Q}_1 (f)(W, M^*_f) \leq \mathbb{Q}_1 (g)(W, M^*_f) \leq \mathbb{Q}_1 (g)(W, M) = \mathbb{Q}_2 (g)(W, M) \]
where the third inequality uses the fact that \( \mathbb{Q}_1 (g) \) is concave, \( M^*_f < M < M^*_g \), and \( M^*_g \) attains the maximum. The second discounting condition is straightforward.

Note that we have focused on the case \( M > \gamma_L \); however, the previous construction also applies to the line with \( M = \gamma_L \) and \( W < \frac{U(\gamma_L) + b}{\gamma_L} \). To complete the construction of \( J \), we derive the value function for the upper-first-best states where \( M = \gamma_L \) and \( W \geq \frac{U(\gamma_L) + b}{\gamma_L} \). Since the agent is risk-neutral, one particular solution has the agent consume \( \frac{1}{\gamma_L} \left( W - \frac{U(\gamma_L) + b}{\gamma_L} \right) \) whenever \( W \geq \frac{U(\gamma_L) + b}{\gamma_L} \); afterwards and the state-pair stays at \( (\frac{U(\gamma_L) + b}{\gamma_L}, \gamma_L) \) without jumps, and the agent obtains \( \frac{b}{\gamma L} < Y \) whenever a cash flow occurs. Based on (21) it is easy to show that in this region
\[ J(W, \gamma_L) = J \left( \frac{U(\gamma_L) + b}{\gamma L}, \gamma_L \right) - \frac{1}{\gamma_L} \left( W - \frac{U(\gamma_L) + b}{r} \right) = \frac{pY}{r} - \frac{w^{-1}(rW)}{r}, \]
which is the first-best result when \( K \), the maximum number of cash flows generated by the agent, is \( \infty \). When \( K \) is finite, we can just replace \( \frac{pY}{r} \) with \( \frac{pY}{r} \left[ 1 - \left( \frac{r}{r + p} \right)^K \right] \) in the above equation.

### A.7 Proof of Proposition 5

The existence of \( J(W, M) \) is established in Section A.6.3. To maximize \( \mathbb{E}_t^* \left[ e^{rt} dG_t \right] \), we need to maximize \( pJ \left( M + \frac{b}{p}, M + \beta_t^M \right) dt + J \left( W + \frac{b}{p}, M + dM_t^I \right) \) (note that \( W = W_{t-} \) and \( M = M_{t-} \)). Since \( J_M(W, M) \geq 0 \), it is without loss of generality to consider two cases: 1) \( \beta_t^M \leq 0 \) and \( dM_t^I = 0 \); and 2) \( \beta_t^M > 0 \) and \( dM_t^I = -\beta_t^M \) \( p dt \). We want to rule out the second case. If it is true, then \( J \left( W, M + dM_t^I \right) = -J_M(W, M) \beta_t^M \) \( p dt \), and we are maximizing a function \( B(\beta_t^M) \) s.t.
\[ B \left( \beta_t^M \right) \equiv J \left( W + \frac{b}{p}, M + \beta_t^M \right) - J_M(W, M) \beta_t^M \] \( p dt \).

It is easy to show that \( B'' \left( \beta_t^M \right) \leq 0 \); then since
\[ B' \left( \beta_t^M \right) \big|_{\beta_t^M = 0} = J_W \left( W + \frac{b}{p}, M \right) - J_M(W, M) < 0, \]
\( B \left( \beta_t^M \right) \) is maximized at \( \beta_t^M = 0 \), contradiction to the second case. Therefore we have shown that the first case holds, i.e., \( dM_t^I = 0 \) and \( \beta_t^M \leq 0 \). Because \( J_M(W, M) = 0 \) for \( M > M^*_t(W) \), the optimal \( \beta_t^M = \min \left( M^*_t(W_{t-} + \frac{b}{p}) - M_{t-}, 0 \right) \).

Now we show that the optimal policy solves the relaxed problem. Our road map is to show that \( \mathbb{E} [G_r(\Pi)] \) which is the investors’ value from any contract \( \Pi \) has an upper bound \( G_0 = J(W_0, M_0) \), i.e., \( \mathbb{E} [G_r(\Pi)] \leq G_0 \); however, under the optimal contract \( \Pi^* \) with policy \( dM_t^D = 0 \) and \( \beta_t^M = \min \left( M^* \left( W_{t-} + \frac{b}{p} \right) - M_{t-}, 0 \right) \), \( \mathbb{E} [G_r(\Pi^*)] = G_0 \).

Given any contract \( \Pi \) that satisfies the necessary conditions to be incentive-compatible and no-savings, we can write the increment of gain process as
\[ dG_t(\Pi) = \mu_G(t) \, dt + e^{-rt} \left[ J \left( W + \frac{b}{p}, M + \beta_t^M \right) - J(W, M) \right] (dN_t - p dt), \]
where one can easily check that due to construction \( \mu_{G(II)}(t) = 0 \) under the optimal policy, and \( \mu_{G(II)}(t) \leq 0 \) for other contracts that satisfies the necessary conditions. And, clearly \( J \left( W + \frac{b}{p}, M + \beta^M_t \right) - J(W, M) \) is bounded (note that \( \beta^M_t \) is bounded as \( M \) is bounded; even in the first-best region where \( W_t \) might be unbounded, \( J \) is linear in \( W \) so \( J \left( W + \frac{b}{p}, M + \beta^M_t \right) - J(W, M) \) is bounded), therefore \( \left\{ e^{-r s} J \left( W_s + \frac{b}{p}, M_s + \beta^M_s \right) - J(W_s - , M_s - \right\} (dN_s - pds) \) forms a well-defined martingale for \( 0 \leq t < \infty \). Because at the termination \( J(W_\tau, M_\tau) = -F_\tau, \mathbb{E}[G_{\tau}(II)] \) is the investors’ payoff. Therefore for any \( t \)

\[
\mathbb{E}[G_{\tau}(II)] = \mathbb{E}\left[G_{t\wedge \tau}(\Pi) + 1_{t \leq \tau} \int_t^\tau e^{-r s} (Y dN_s - c_s) ds - e^{-r \tau} F_\tau \right] 
\leq G_0 + e^{-r t} \mathbb{E}\left[ \int_t^\infty e^{-r(s-t)} Y dN_s \right].
\]

where \( \mathbb{E}\left[ \int_t^\infty e^{-r(s-t)} Y dN_s \right] \) represents the present value of firm’s total cash flow (without early termination), which is bounded. Therefore when \( t \to \infty, \mathbb{E}[G_{\tau}(II)] \leq G_0 \) for any contract; while under the optimal contract with policy \( dM_t^D = 0 \) and \( \beta^M_t = \min \left( M^* \left( W_{t^-} + \frac{b}{p} \right) - M_{t^-}, 0 \right), \mu_{G(II)}(t) = 0 \) implies that the inequality in (42) holds in equality, therefore \( \mathbb{E}[G_{\tau}(II)] = G_0 \). This shows our claim.

**A.7.1 Proof of Theorem 1**

The proof is essentially the combination of Proposition 5 and the argument right before Theorem 1. The first-best result directly follows from Section 4.5. We have the following lemma to show formally that under the downward-rigid wage contract the agent is optimal to work and consume the wage.

**Lemma 6** Suppose that the agent has an hypothetical saving of \( S_0 \geq 0 \). Then the agent’s optimal value when facing the downward-rigid wage contract with a state-variable pair \( (W, M) \), is

\[
V(M, W, S) = W - \Phi(M, 0) + \Phi(M, S),
\]

where

\[
\Phi(M, S) = \frac{1 - e^{-\gamma (c(M) + r S)}}{r} = \frac{1}{r e^{-\gamma r s}}. \]

**Proof.** Simple algebra yields \( V_M(M, W, S) = \frac{1}{r} \left( 1 - e^{-\gamma r S} \right) \geq 0 \), and \( V_S(M, W, S) = M e^{-\gamma r S} \). Introduce the agent’s auxiliary gain process as

\[
G_t^A = \int_0^t e^{-r s} \left( U(\tilde{M}_s) + b \left( 1 - \frac{\tilde{a}_s}{p} \right) \right) ds + e^{-r t} V(M_{t^-}, W_{t^-}, S_t).
\]

where the evolutions of state variables are

\[
\begin{align*}
    dS &= r S dt + c(M) dt - c(\tilde{M}) dt, \\
    dM &= \beta^M dN_t, \\
    dW &= (r W - U(M) - b) dt + \frac{b}{p} dN_t(\tilde{a}),
\end{align*}
\]

and we use the actual marginal utility \( \tilde{M}_s \) as the agent’s control variable. It does not make a difference by using \( S_t \) or \( S_{t^-} \) as \( S \) has continuous pathes. Then

\[
\mathbb{E}_t\left[ e^{r t} dG_t^A \right] = U(\tilde{M}) dt + b \left( 1 - \frac{\tilde{a}}{p} \right) dt - r V dt + dW + V_M(M, W, S) dM + V_S(M, W, S) dS
\]

It is easy to see that \( \tilde{a} = 0 \) maximizes \( \mathbb{E}_t\left[ e^{r t} dG_t^A \right] \) (and strictly so when \( \beta^M_t < 0 \) and \( S > 0 \); when \( S = 0, \tilde{a} = p \) is also optimal—this is the optimal policy along the equilibrium path). Then we have

\[
\mathbb{E}_t\left[ e^{r t} dG_t^A \right] \leq U(\tilde{M}) + r W - U(M) - r V + M e^{-\gamma r S} \left( r S + c(M) - c(\tilde{M}) \right)
\]

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The FOC of $\hat{M}$ (recall the definition of $U(\cdot)$ in Eq. (16) and $c(\cdot)$ in Eq. (17)) yields that (we can also easily check that FOC is sufficient because RHS is concave in $\hat{M}$)

$$\hat{M} = M e^{-\gamma r S}.$$ Plugging in, we have $E_t - \left[ e^{r t} dG_t \right] / dt \leq 0$. Because the inequality could hold in equality when the optimal policy is used, standard verification argument similar to the proof of Proposition 5 shows our claim. 

Given this lemma, the agent’s value without saving $S_0 = 0$ is just $W$, which is achieved by working and no saving. This shows our claim.

### A.8 Appendix for Section 4

#### A.8.1 Appendix for Section 5.3

Recall that by implementing the myopic effort, investors suffer a non-contractible loss $\Delta$. Therefore we have (recall that $\bar{p} = p + \epsilon$),

$$E_t - \left[ e^{r t} dG_t \right] = \left[ \begin{array}{c}
-r J - c(M) + \bar{p} \left( Y + J (W + \beta_t^W, M + \beta_t^M) - J (W, M) \right) \\
+J_W \left( r W - U (M) + \frac{b}{p} - \beta_t^W \bar{p} \right) 
\end{array} \right] dt + J_M (W, M) \bar{p} (M - \gamma_L) dt - \Delta dt$$

We want to give an upper-bound estimate for $E_t - \left[ e^{r t} dG_t \right]$ given the condition $dM_t^D \leq -\beta_t^M \bar{p} dt$ and $\beta_t^W \geq \frac{b}{\bar{p}}$.

Recall that $dM_t^D \leq -\beta_t^M \bar{p} dt$; then similar to the first paragraph in the proof of Proposition 5 in Appendix A.7, we can show that setting $dM_t^D = -\beta_t^M \bar{p} dt$ and choosing the lowest (most negative) $\beta_t^M$ maximizes $E_t - \left[ e^{r t} dG_t \right]$. The lowest possible $\beta_t^M$ is $\gamma_L - M_t \leq 0$. Therefore we have

$$E_t - \left[ e^{r t} dG_t \right] \leq \left[ \begin{array}{c}
-r J - c(M) + \bar{p} \left( Y + \frac{J (W + \beta_t^W, \gamma_L) - J (W, M)}{\beta_t^M \bar{p}} \right) \\
+J_W \left( r W - U (M) + \frac{b}{p} - \beta_t^W \bar{p} \right) 
\end{array} \right] dt + J_M (W, M) \bar{p} (M - \gamma_L) dt - \Delta dt$$

where the second inequality is due to $J_M \geq 0$. Now the only choice variale is $\beta_t^W$; because $J$ is concave,

$$\max_{\beta_t^W \geq \frac{b}{\bar{p}}} \bar{p} J \left( W + \beta_t^W, M \right) - J_W \beta_t^W \bar{p}$$

yields a solution of $\beta_t^W = \frac{b}{\bar{p}}$. Therefore

$$E_t - \left[ e^{r t} dG_t \right] \leq \left[ \begin{array}{c}
-r J - c(M) + \bar{p} \left( Y + \frac{J (W + \frac{b}{\bar{p}}, M) - J}{\beta_t^W \bar{p}} \right) \\
+J_W \left( r W - U (M) - b \right) 
\end{array} \right] dt + J_M (W, M) \bar{p} (M - \gamma_L) dt - \Delta dt$$

We take $\epsilon$ to be arbitrarily small. Because $J_{W,M} < 0$, when $M$ is fixed, $J_M$ attains the maximum when $W = \frac{U(M)}{r}$. Therefore

$$\Delta > \max_{M \in [\gamma_L, \gamma]} p (M - \gamma_L) J_M \left( \frac{U(M)}{r}, M \right).$$
A.8.2 Proof of Proposition 6

Take the auxiliary gain process $G_t$ as defined in 27. In Section 5.3.1 and Section 5.3.2, we have shown that whenever actions other than working are implemented,

$$dG_t = \mu_G(t) \, dt + e^{-r t} \left[ J \left( W + \beta_t^W, M + \beta_t^M \right) - J (W, M) \right] (dN_t (a_t) - a_t dt)$$

where $\mu_G(t) \leq 0$. We require $\beta_t^W$ to be bounded in any feasible contract; because $\beta_t^M$ has to be bounded since $M$ is bounded, then $J \left( W + \beta_t^W, M + \beta_t^M \right) - J (W, M)$ is bounded (even in the first-best region where $W$ might be unbounded—see the argument in the proof of Proposition 5 in Appendix A.7). Consequently,

$$\left\{ \int_0^t e^{-r s} \left[ J \left( W_{s-} + \frac{b}{p}, M_{s-} + \beta_s^M \right) - J (W_{s-}, M_{s-}) \right] (dN_s - pds) \right\}$$

forms a well-defined martingale for $0 \leq t < \infty$. We then can invoke the same argument as in the proof of Proposition 5 to show that the contract given in Theorem 1 (which implements working always) is optimal among all contracts that may implement other actions.

A.9 Appendix for Section 6.1

Following Sannikov (2008), we denote the investors’ concave value function as $f(W)$, and continuation payoff $W$ follows

$$dW = (r W_t - u(c_t)) \, dt + \frac{b}{p} (dN_t - pdt),$$

where $c^*$ solves the investors’ HJB equation

$$rf(W) = \max_{c \geq 0} \left\{ pY - c + p \left[ f \left( W + \frac{b}{p} \right) - f(W) \right] + f'(W) [r W - u(c) - b] \right\}. \quad (44)$$

Clearly, due to the risk-neutrality for a sufficiently high consumption level, similar to the previous discussion there is an absorbing first-best state for $W \geq \frac{U(r_l + b)}{p}$, and $f'(W) = \frac{1}{p}$. Note that in Sannikov (2008) the upper-absorbing state corresponds to the case where the wealth effect becomes extreme, and the firm is terminated. The difference is purely due to different utility specifications.

In the lower region where $f'(W) > -\frac{1}{p}$, it is easy to show that the optimal wage policy, as a function of $W$, is

$$c^* = \left\{ \begin{array}{ll}
\frac{1}{r} \ln \left( \frac{W}{r W + 1} \right) & \text{when } f'(W) < -\frac{1}{p} \\
0 & \text{otherwise}
\end{array} \right. \quad .$$

This policy can be understood as follows. In (44), paying one more dollar of wage has a unit marginal cost; and on the benefit side, it reduces the agent’s continuation payoff by $u'(c)$, so the marginal benefit is $-f'(W) u'(c)$. The above policy equates the marginal cost with the marginal benefit whenever possible. As $f$ is concave, $c^*$ will bind at zero for low $W$’s, which reflects the fact that when the inefficient termination (once $W = 0$) is close, the marginal benefit of reducing continuation payoff $-f'(W) u'(c)$ either is small, or even becomes negative.

A.10 Appendix for Section 6.3

Let $g(\cdot) = u'(\cdot)$; then $U(M) = u \left( g^{-1}(M) \right)$, and $l(M) = \frac{U(M)}{r}$. It is easy to check that

$$l'(m) = \frac{m}{r u''} < 0, \quad l''(m) = \frac{u'' - u'''}{r (-u''')^2} > 0,$$
where the second inequality, which is equivalent to $u'' > \left(\frac{u'}{u}\right)^2 > 0$, is imposed as assumption. Here $J^L \left(\frac{U(M)}{r}\right) = -\frac{q^{-1}(M)}{r}$; so $\frac{dJ^L}{dM} = \frac{-1}{r\left(-u^\prime\prime\right)} > 0$ and $\frac{d^2J^L}{dM^2} = \frac{1}{r^2\left(ww''\prime\right)} < 0$. Similar to the previous analysis, define a nonlinear transformation (omitting the superscript $i$, if any)

$$J(W, M) = j \left(W - \frac{U(M)}{r}\right) = j(w, m).$$

Therefore

$$J_W = j_w, J_M = \frac{m}{r(-u^\prime)} j_w + jm$$

and

$$J_{WW} = j_{ww}, J_{WM} = \frac{m}{r(-u^\prime)} j_{ww} + j_{wm}, J_{MM} = -j_{ww}'' + j_{ww} \left(1\right)^2 - 2t' j_{wm} + j_{mm}$$

and

$$J_{WW} J_{MM} - (J_{WM})^2 = -j_{ww} j_{ww}'' + j_{ww} j_{mm} - (j_{wm})^2.$$

The following two lemma give corresponding results for Proposition 4 and Proposition 3.

**Lemma 7** For the setting-wage stage value function $j^{i-1}(w, m)$, we have

1. $j_{w}^{i-1} \leq -\frac{1}{\gamma_L}$, and $\frac{1}{r\left(-u^\prime\prime\right)} < j_{m}^{i-1} \leq \frac{m}{r\left(-u^\prime\prime\right)\gamma_L}$;

2. $j_{WW}^{i-1} < 0$, $j_{MM}^{i-1} \leq 0$, and $j_{WW}^{i-1} j_{MM}^{i-1} - \left(j_{WM}^{i-1}\right)^2 \geq 0$. It implies that $J^{i-1}$ is concave;

3. $J_{WW}^{i-1} = \frac{m}{r(-u^\prime)} j_{ww}^{i-1} + j_{wm} \leq 0$, and $J_{MM}^{i-1} = \frac{m}{r(-u^\prime)} j_{ww}^{i-1} + j_{mm} \geq 0$. We have $J_{MM}^{i-1} \left(\frac{U(M)+b}{r}\right), M) = 0$.

**Lemma 8** For the production stage value function $\tilde{J}^i(w, m)$, we have

1. $\tilde{j}_w \geq -\frac{1}{\gamma_L}$ and $\frac{1}{r\left(-u^\prime\prime\right)} < \tilde{j}_m \leq \frac{m}{r\left(-u^\prime\prime\right)\gamma_L}$;

2. $\tilde{j}_{WW} < 0$, $\tilde{j}_{MM} < 0$, and $\tilde{j}_{WW} \tilde{j}_{MM} - \left(\tilde{j}_{WM}\right)^2 > 0$. It implies that $\tilde{J}^i$ is concave;

3. $\tilde{J}_{WM} = \frac{m}{r(-u^\prime)} \tilde{j}_{ww} + \tilde{j}_{wm} < 0$. We have $\tilde{J}_M \left(\frac{U(M)+b}{r}\right), M) < 0$.

For detailed proofs, and other requirements when some singularity is involved (e.g., the case where $u'' \rightarrow 0$ when $u' \rightarrow \gamma_L$, rather than the case where there exists $\pi$ such that $u''(\pi) < 0$ and $u''(\pi) = 0$ as assumed in the main text), see He (2007).

### A.11 Appendix for Section 6.4

We again construct $J^{RP}(W, M)$ recursively. The following lemma lists the properties of $j^{RP,i-1}(W, M)$. In property 4, $w^{i-1}(m)$ is the renegotiation curve discussed in the main text, and $W^{RP,i-1,\ast}(m)$ is the wage-setting curve similar to the definition in (24).

**Lemma 9** For the stage value function $j^{RP,i-1}(w, m)$, we have the following properties:

1. $-\frac{1}{\gamma_L} \leq j_W^{RP,i-1} \leq 0$, $j_M^{RP,i-1} > \frac{1}{\gamma_M}$, and $0 \leq \frac{1}{\gamma_M} j_W^{RP,i-1} + j_M^{RP,i-1} \leq \frac{1}{\gamma_M}$.

2. $\frac{R^{RP,i-1}}{J_{ww}} < 0$, $J_{mm}^{i-1} < 0$, $J_{wm}^{i-1} > 0$, and $J_{ww}^{i-1} J_{mm}^{i-1} - \left(J_{wm}^{i-1}\right)^2 \geq 0$. Therefore $j^{RP,i-1}(w, m)$ is concave.

3. $\frac{1}{\gamma_W} J_{ww}^{i-1} + j_{ww}^{i-1} \leq 0$, $\frac{1}{\gamma_M} J_{ww}^{i-1} \left(\frac{b}{r}, m\right) + j_{ww}^{i-1} \left(\frac{b}{r}, m\right) \geq 0$, and $\frac{1}{\gamma_W} J_{ww}^{i-1} \left(\frac{b}{r}, m\right) + j_{ww}^{i-1} \left(\frac{b}{r}, m\right) = 0$. 

45
Consider the production stage in \(i^{th}\) subperiod. There exists a curve \(w^i(m)\) such that \(\tilde{R}^{P,i}\) takes the value \(J^L(\frac{U(m)}{r})\), and \(\tilde{R}^{P,i}\) = 0 on this curve. Similar to (35), one can check that

\[
\tilde{R}^{P,i}(w, m) = p[b - ru]^2 \left\{ \int_{w(m)}^{w} \frac{\tilde{R}^{P,i-1}(w, m) - J^L(\frac{U(m)}{r})}{(b - ru)^2} \, dx + \frac{\tilde{Y} + J^{P,i-1} + J^L(\frac{U(m)}{r}) - J^L(\frac{U(m)}{r})}{(b - ru)^2} \right\},
\]

where \(\tilde{Y} = \frac{pY - rL}{r}\). In the spirit of renegotiation-proof contract, at \(w^i(m), \tilde{R}^{P,i} = p \left[ \tilde{Y} + J^{P,i-1} + J^L(\frac{U(m)}{r}) \right]\) is zero at \(w^i(m)\). Therefore we define

\[
w^i(m) = \inf \left\{ 0 \leq x \leq \frac{b}{r} : \tilde{Y} + J^{P,i-1} + J^L(\frac{U(m)}{r}) = 0 \right\}. \tag{45}
\]

We assume that \(w^i(m)\) defined in (45) satisfies \(w^i(m) < \frac{b}{r}\). Because \(j\) is decreasing in \(x\), this condition holds when \(L\) is relatively large so that \(\tilde{Y} = \frac{pY - rL}{r}\) is relatively small. Under this condition, we can show that \(w^i(m) < W^{RP,i-1} - \frac{U(m)}{r}\). For instance, when \(M = \gamma_L\), for \(W^i(\gamma_L) = J^L(\frac{U(m)}{r})\), we require that the investors’ value at termination is greater than their value at the upper-first-best boundary point, i.e., \(J^L(\frac{U(m)}{r}) \geq \frac{J^L(\frac{U(m)}{r})}{r} < \frac{b}{r}\). We have the following lemma for \(\tilde{R}^{P,i}\).

**Lemma 10** For the production-stage value function \(\tilde{R}^{P,i}(w, m)\), we have

1. \(\tilde{R}^{P,i} < 0, \tilde{R}^{P,i} > \frac{1}{\gamma_m} \) and \(\tilde{R}^{P,i} - \tilde{R}^{P,i} < \frac{1}{\gamma_m}\).
2. \(\tilde{R}^{P,i} < 0, \tilde{R}^{P,i} > 0, \tilde{R}^{P,i} > \frac{1}{\gamma_m}\) and \(\tilde{R}^{P,i} - \tilde{R}^{P,i} \geq \frac{1}{\gamma_m} < 0\). Therefore \(\tilde{R}^{P,i}(w, m)\) is concave.
3. \(\frac{1}{\gamma_m} \tilde{R}^{P,i} + \tilde{R}^{P,i} \leq 0, \frac{1}{\gamma_m} \tilde{R}^{P,i} \left( \frac{1}{r}, m \right) + \tilde{R}^{P,i} \left( \frac{1}{r}, m \right) < 0\).
4. \(w^i(m) \geq 0\).

For detailed proofs, see He (2007). When \(L\) is small (for instance, \(L = 0\), \(w^i(m)\) and \(W^{RP,i}(\gamma_L) - \frac{U(m)}{r}\) both bind at \(\frac{b}{r}\). At this point, without success the agent stays at that point, and after a jump the agent is promoted to another point with a lower \(m\) (higher wages). Because the termination is extremely inefficient (\(pY > \frac{b}{r}\)) so keeping the project alive is better off always), termination will be off-equilibrium.

### A.12 Proof of Proposition 7

Since \(dQ\) information does not add value in implementing \(a = 0 \) or \(a = r\), both actions are suboptimal. In implementing \(a = p\), recall that

\[
dW_t = (rW_t - U(M_t^i)) \, dt + \beta^W_t (dN_t - pdt) + x_t (dQ_t - pdt),
\]

where \(\beta^W_t = \frac{b}{r} + k_t > \frac{b}{r}\), and \(x_t \geq \frac{k_t}{\Delta}\). Note that \(k_t \geq 0\); otherwise the agent shirks (without affecting the \(dQ_t\) performance). Suppose that the evolution of \(M\) can be written as,

\[
dM_t = M_t dt + \left( M_t^0 - M_t^i \right) + x_t M_t dZ_t;
\]

any drift is absorbed in the term of \(M_t^0 - M_t\). Since \(J\) is concave, to mitigate the second-order effect \(\frac{k_t^2 \sigma^2}{\Delta^2} J_{WW} + 2 \frac{k_t^2 \sigma^2}{\Delta^2} J_{WM} + \left( \frac{x_t}{\Delta} \right)^2 J_{MM}\), the optimal \(x_t\) is \(- \frac{k_t \sigma J_{WW}}{\Delta^2 J_{MM}}\), which gives a lower bound estimate of the second order loss due to extra loadings on \(dQ\) information:

\[
\frac{k_t^2 \sigma^2}{\Delta^2} \left( J_{WW} J_{MM} - J_{WM}^2 \right) = \frac{k_t^2 \sigma^2}{\Delta^2} \left( \frac{1}{\gamma_m} J_{WW} + \frac{2}{\gamma_m} J_{WM} + J_{MM} \right).
\]
Therefore, on these states, since the domain of integration in $J$ is close enough to the upper-first-best region ($M$ performance is bounded away from zero (the claim for $K$ by $\gamma$ where $M$ is the gain). We ignore $A$ for later, then it must be the case that $A = 0$.

Now for investors, the value still can be written as $J^{Q} = \mathbb{E} [G_{\gamma}]$ with $G_{\gamma}$ defined in (27), under some optimal contract that incorporates both the $Y$ and $Q$ information. As $G_{0} = J$, $J^{Q} - J = \mathbb{E} [G_{\gamma} - G_{0}]$, the net gain by incorporating $Q$ performance is

$$J^{Q} - J = \mathbb{E} \left\{ \int_{0}^{T} e^{-rt} \left[ -rJ - c(M_{t^{-}}) + p \left( Y + \left[ J \left( W + \frac{b}{p}, M_{t^{-}} \right) - J \right] + J_{W} (rW - U(M) - b) \right) \right] dt \right\}$$

$$= \mathbb{E} \left\{ \int_{0}^{T} e^{-rt} \left[ -rJ - c(M_{t^{-}}) + p \left( Y + \left[ J \left( W + \frac{b}{p}, M_{t^{-}} \right) - J \right] + J_{W} (rW - U(M) - b) \right) \right] dt \right\} + e^{-rt} \left( \frac{b^{2} r^{2}}{2} \right) \left( \frac{J_{MM} M_{t^{-}} - J_{WM_{t^{-}}}^{2}}{J_{MM}} \right) dt$$

$$= \mathbb{E} \left\{ \left( J_{MM} M_{t^{-}} - J_{WM_{t^{-}}}^{2} \right) dt \right\}, \quad (46)$$

where $\gamma$ could take the value $\infty$, and keep in mind that $M = M_{t^{-}}$.

Our goal is to show that when $\sigma \to \infty$, $J^{Q} - J \leq \eta(\sigma)$ where $\eta(\sigma) \to 0$. Due to construction, the first line is zero. Now we analyze the other two terms. Clearly, the second term can be positive, while the third term is negative (due to concavity of $J$); and the trade-off is between these two terms.

In light of this trade-off, we will only focus on the region where this trade-off matters. On the state space, when $(W, M)$ is close enough to the upper-first-best region ($M = \gamma_{L}$), $J$ fails to be strictly concave in the first-best region $W \geq \frac{U\gamma_{L} + b}{\gamma r}$. We have shown that in (29) that we have

$$M_{t}^{0} - M \leq A (M - \gamma_{L}) dt. \quad (47)$$

Therefore, on these states, since $M_{t}^{0} - M$ has to be zero in light of (47), the gain is also zero. To tackle this issue, we decompose the domain of integration in $J^{Q}$ into three parts based on the $(W, M)$ space. All the following analysis will be on the set of

$$A^{1} (\varepsilon) = \left\{ (W, M) : W < \frac{U\gamma_{L} + b}{\gamma r} - \varepsilon, M > \gamma_{L}, \text{and } J_{M} > 0 \right\} \quad (48)$$

where $\varepsilon$ is arbitrarily small. Using the results in Section A.6, one can check that on $A^{1} (\varepsilon)$, $J_{MM} M_{t} - J_{WM_{t}}^{2} = \frac{J_{MM} J_{WM_{t}} - J_{WM_{t}}^{2}}{J_{MM}}$ is strictly negative and bounded by $\phi(\varepsilon) < 0$. The other two sets are

$$A^{2} = \left\{ (W, M) : M = \gamma_{L} \text{ and } J_{M} = 0 \right\} \text{, } A^{3} (\varepsilon) = \left\{ (W, M) : \frac{U\gamma_{L} + b}{\gamma r} - \varepsilon < W < \frac{U\gamma_{L} + b}{\gamma r}, M > \gamma_{L} \right\}.$$ 

We ignore $A^{2}$ as the gain is zero. For $A^{3} (\varepsilon)$, one can show that $M^{*} \left( \frac{U\gamma_{L} + b}{\gamma r} \right) = 0$, so on $A^{3} (\varepsilon)$ we only have to consider the region $M \geq \gamma_{L} - y(\varepsilon) M$, where $y(\varepsilon) \to 0$ as $\varepsilon \to 0$. This implies that around this singular point the gain on $A^{3}$ is bounded by $\varepsilon$, and by taking $\varepsilon \to 0$, we can focus on the region $A^{1} (\varepsilon)$ later on.

Our road map is to first give an expression of the positive gain in Step 1. Then we show in Step 2 that when $\sigma \to \infty$, in order for $J^{Q} - J$ to be potentially positive, the loading $k$ has to vanish, i.e., $k(\sigma) \to 0$. Then in Step 3 we come back to the result in Step 1, and show that given $k(\sigma) \to 0$ the gain in the second line goes to zero as well.

• Step 1. For the second line, let $M_{t}^{0} - M = xdt$ (we can easily show it must be in the order of $dt$), which is $dM_{t}^{D}$ in the text. Since $pM_{t}^{D} \leq M - (1 - pdt) M_{t}^{0}$ to prevent saving along the equilibrium working path, setting $M_{t}^{0} = \frac{\gamma}{p}$.

32To see this, similar to the argument of Cauchy-Schwartz inequality in A.6, one can show that

$$\frac{\gamma}{J_{ww} J_{wm}} - (\frac{J_{wm}}{J_{ww}})^{2} > -\frac{\gamma}{J_{ww}^{2}} \left( \frac{p}{r + p} - K(m) \left( b - rw \right) \right) \left( \frac{b - rw}{b} \right)^{1 + \frac{\gamma}{r + p}} - \frac{r}{\gamma r_{m}^{2}} - \frac{1}{\gamma r_{m}^{2}}$$

where $K(m) = \frac{1}{\gamma r_{m}^{2}} \left[ \frac{m}{2} - 1 \right]^{2} < 1$. The above term takes a positive bounded-from-below value for $w \leq \omega^{*}(m) < \frac{b}{p}$, because each term is bounded away from zero (the claim for $-\frac{\gamma}{J_{ww}}$ is due to (39)). Also it follows that $J_{MM} = \frac{1}{\gamma} J_{ww} + 2 \frac{\gamma}{\gamma r} J_{wm} + \frac{\gamma}{\gamma r} J_{mm} < 0$, because if it is zero, then it must be the case that $\frac{\gamma}{J_{ww}} \geq \frac{\gamma}{\gamma r} \sqrt{J_{wm} J_{mm}}$, contradiction.
maximizes the second line. Then the second line is dominated by

\[
p \left[ J \left( W + \frac{b}{p}, M - \frac{x}{p} \right) - J \left( W + \frac{b}{p}, M \right) \right] dt + J_M \cdot x dt
\]

because \( J \) is concave in \( M \). Note that \( J_M \left( W, M \right) - J_M \left( W + \frac{b}{p}, M \right) > 0 \); it just captures that idea that investors gains by reducing wage without jump (so \( M_1^0 > M \)). For a simpler notation, later we replace this gain in (49) by

\[
J_M \left( W, M \right) \left( M_1^0 - M \right),
\]

because its analytical property is exactly the same as the one in (49). We will bound this gain in Step 3.

- Step 2. The third line is the cost brought on by \( dQ \). Our main argument will be that since \( J \) is concave (\( \frac{J_{WW} \cdot J_{MM} - J_{WM}^2}{J_{MM}^2} \) is negative), when \( \sigma^2 \to \infty, k_t \) has to be sufficiently small. Because then the shirking loss is diminishing, this limits the magnitude of \( M_1^0 - M \), and therefore the contractual gain identified above goes to zero.

Denote by \( \nu \) the measure induced by \( E \left[ \int_0^t e^{-\nu t} dt \right] \) on \( A^1(\epsilon) \) defined in (48). Because \( \int A^1(\epsilon) \, d\nu \leq \frac{1}{\gamma} \), Holder’s inequality yields (recall that \( \phi(\epsilon) < 0 \))

\[
\int A^1(\epsilon) k_t^2 \sigma^2 \left( \frac{J_{WW} \cdot J_{MM} - J_{WM}^2}{J_{MM}} \right) d\nu < \sigma^2 \phi(\epsilon) \int A^1(\epsilon) k_t^2 d\nu < \sigma^2 \phi(\epsilon) \left( \frac{\int A^1(\epsilon) k_t d\nu}{\int A^1(\epsilon) d\nu} \right)^2 < r \sigma^2 \phi(\epsilon) \left( \int A^1(\epsilon) k_t d\nu \right)^2.
\]

Now because the total expected gain from a complete contract is bounded by the difference between investors’ value with and without private savings, to ensure that \( J^{Q} - J > 0 \) we must have \( \int A^1(\epsilon) k_t d\nu \to 0 \) when \( \sigma \to \infty \). Therefore, \( k_t \to 0 \) except on a zero measure set.

- Step 3. Based on the above argument, we change our interpretation which relaxes the problem. Essentially, writing \( dQ \) into the contract creates a convex cost structure for the agent to exert \( a = \gamma \) relative to \( a = p \), and investors can specify \( k_t > 0 \) without inducing \( a = \gamma \). When \( \sigma \to \infty \), the above result suggests that the convexity (captured by the upper bound of \( k_t \)) diminishes. Therefore, we now consider the model with a convex effort cost as a modification of the linear cost specification, where the agent incurs a cost \( (\frac{b}{p} + k) \epsilon \) in taking \( a = \gamma \), where \( k > 0 \) is a constant. We then investigate the limiting case where \( k \to 0 \), i.e., effort cost becomes linear. This relaxes the investors’ problem without any cost (instead of using \( dQ \), the convexity comes for free); but we will show that investors’ potential gain \( \int A^1(\epsilon) J_M \left( M_1^0 - M \right) d\nu \) goes to zero even in this framework.

Based on this relaxation, we restrict attention to the case where \( W \) and \( W \) are deterministic (as investors no long need \( dQ \) performance). Because the agent’s saving motive is determined by expected marginal utility, and the investors’ value function is concave, this treatment is innocuous. In fact, we can show that, randomizations of \( M \) and \( W \) in fact increase the agent’s private savings gain, and in the same time reduces the investors’ value—therefore we obtain a stronger result by ruling out randomization. Interested readers can find the proof in He (2007).

Note that the agent’s incentive of “shirking and saving” is counteracted by \( k > 0 \). When \( k \to 0 \), to prevent “shirking and saving,” the private savings gain must converge to zero also. More specifically, fix a starting time \( 0 \) and focus on the path without jumps. Suppose the initial marginal utility is \( M_0 \) (so consumption level is \( c_0 \)). Private savings gain going to zero implies that, future consumption without jumps cannot fall below \( c_0 - \eta \), for some positive \( \eta \) (note that (47) implies that the consumption falling path must be continuous with a bounded speed \( M_t^0 - M_t^- < p \left( \gamma - \gamma_L \right) dt \), which ensures a strictly positive consumption smoothing gain once \( c_t \) goes under \( c_0 \) strictly). Equivalently, for \( \forall t_2 > t_1 \), \( M_{t_2} < M_{t_1} + \eta \) for some positive \( \eta \) almost surely, where \( \eta (\sigma) \to 0 \) as \( k (\gamma) \to 0 \).
Because $J_M \geq 0$, w.l.o.g we restrict $\{M_t\}$ to lie within a band with a width $\eta$, i.e., $M_t \in [M_0, M_0 + \eta]$.

Denote the gain from time 0 to $t > 0$ as $H_t$. Let $\tilde{r} = r + p$, and focus on the path before any jump occurs (later on we will integrate over all jumps). We have

$$
H_t = \mathbb{E} \int_0^t e^{-\tilde{r}s} [J(W_{s-}, M_s) - J(W_{s-}, M_{s-})] 1_{\{N_s = 0\}} = \int_0^t e^{-\tilde{r}s} [J(W_{s-}, M_s) - J(W_{s-}, M_{s-})]
$$

Due to (47), there is no upward jump for $K$. Setup implies that the gain is dominated by $dW$ and these constants where the first equality uses the Intermediate Value Theorem with $f$ and bounded $J_{WM}$, we can find $K_1$ so that $\int_0^t e^{-\tilde{r}s} J_{WM} (W_s, \tilde{M}) dW_s \cdot (M_s - M_0) \geq -K_1 \cdot \eta$. Therefore,

$$
H_t \leq e^{-\tilde{r}t} J(W_t, M_t) - e^{-\tilde{r}t} J(W_0, M_0) + \tilde{r} \int_0^t J(W_s, M_s) e^{-\tilde{r}s} ds - \tilde{r} \int_0^t J(W_s, M_0) e^{-\tilde{r}s} ds + K_1 \eta
$$

and these constants $K_1$’s are independent of $t$, $W$, and $M$. Now integrating over all possible jumps, standard Poisson setup implies that the gain is dominated by $K \eta$ where $K$ is a constant.

As a summary, when $\sigma \to \infty$, the convexity $k(\sigma)$ that is allowed in the contract goes to zero, which implies that $\eta(\sigma)$, the increment of $M$ without a jump allowed in the complete contract, goes to zero. Therefore, as shown above, the contractual gain in (46) converges to zero. This shows that even though it will be beneficial to incorporate $dQ$ information, when its precision drops to zero which is featured by most soft information, the value from the optimal complete contract converges to $J$ derived in the main text.

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33 Due to (47), there is no upward jump for $\{M_t\}$, but it might exhibit downward jump. However, $\{M_t\}$ has bounded variation for the existence of the integral. Also, as $J_M > 0$, the suicide strategy of $M_t = M_t^* + f(t)$ where $M_t^* \in [M_0, M_0 + \eta]$, $f(0) = 0$ and $f'(t) < 0$ is suboptimal. An argument similar to the one presented here can be invoked to rigorously show this claim (see He (2007)).