Price Discovery in Waiting Lists:
A Connection to Stochastic Gradient Descent

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Price Discovery in Waiting Lists

Waiting times serve as prices in waiting lists
- Agents choose among items and associated waiting times
- Can be similar to standard competitive equilibria

Waiting list mechanisms are commonly used
- e.g., public housing, ride sharing, …

Natural price discovery process
- Planner does not set prices
- Prices determined by endogenous queue lengths
- Prices adjust with each arrival
  - Similar to Tâtonnement – price increases with demand (agents join queue), decreases with supply (items arrive)
Example – Queueing for One Item

- Single item, arrives at Poisson rate 1
- Agents arrive at Poisson rate 2
  - Agents observe the queue length, can join the queue or leave
  - Quasilinear utility
    \[ v = 0.02 \cdot w \]
    with \( v \sim U[0,1] \) i.i.d.

**Static benchmark:**
- Collect all items and agents that arrive until (large) time \( T \)
- Assigning agents if \( v \geq 1/2 \) maximizes allocative efficiency
- Market clearing price is \( p^* = 1/2 \)
Example – One Item

Probability in steady state

Queue Length
Example – One Item

The graph shows the relationship between the agent's value and the probability of assignment. The blue line represents the assignment probability, while the black line represents the optimal assignment probability. As the agent's value increases, the probability of assignment also increases, approaching 100% for optimal assignment.
Price Discovery in Waiting Lists

**Question:** Allocative efficiency under fluctuating prices

**Main Result:** Loss from price fluctuations is bounded by the *adjustment size*

- Bound is (almost) tight
- Conditions for when the loss is negligible

**Methodological contribution:**

- Price adaptation as a stochastic gradient decent (SGD)
- Duality, Lyapunov functions

**Price rigidity:** tradeoff between learning speed and overreaction
Related Work

Dynamic matching mechanisms:


Convergence of tâtonnement processes using gradient descent:


Cost of fluctuations:

Model

**Items:** Arrive according to Poisson process, total rate $\mu = 1$
- Finite number of items $J_\emptyset = \{1,2,...,J\} \cup \{\emptyset\}$
- With probability $\mu_j$ arriving item is of type $j$

**Agents:** Arrive according to Poisson process with total rate $\lambda$
- Agent type $\theta \in \Theta$, drawn i.i.d. according to distribution $F$
- Possibly uncountably many or finitely many types

**Quasi-Linear Utility:**
- $u_\theta(j, w)$ is the utility of type $\theta$ agent assigned item $j$ with wait $w$
  $$u_\theta(j, w) = \nu(\theta, j) - c(w)$$
- Agents can leave immediately (balk) to obtain utility $\nu(\theta, \emptyset) = 0$
- Values are private information
- $\nu(\theta, j)$ is bounded; $c(\cdot)$ is smooth, strictly increasing and convex or concave
Assignments and Allocative Efficiency

Assignments $\eta$

Let $\eta_t \in J_\emptyset$ denote the item assigned to agent who arrived at $t$.

Allocative efficiency

$$W(\eta) = \liminf_{T \to \infty} \frac{1}{|A_T|} \sum_{t \in A_T} v(\theta_t, \eta_t)$$

Optimal allocative efficiency

$$W^{OPT} = \mathbb{E} \left[ \sup_{\eta} W(\eta) \right]$$

- Restricting attention to assignments $\eta$ that satisfy a no-Ponzi condition.
The Waiting List Mechanism

Separate queue for each item $j \in J$
- First Come First Served (FCFS) assignment policy
- Agents who join a queue wait until assigned (no reneging)

Choice of agent $\theta$ who observes $q$:

$$a(\theta, q) = \arg\max_{j \in \mathcal{J} \cup \{\emptyset\}} \left\{ v(\theta, j) - \mathbb{E}[c(w_j) | q] \right\}$$

- Observes all queue lengths $q = (q_1, \ldots, q_J)$
- Can join any queue, or leave unassigned

- Simplified version of public housing assignment
The Waiting List Mechanism

Separate queue for each item $j \in J$
- First Come First Served (FCFS) assignment policy
- Agents who join a queue wait until assigned (no reneging)

Choice of agent $\theta$ who observes $q$:

$$a(\theta, q) = \underset{j \in J \cup \{\emptyset\}}{\text{argmax}} \left\{ v(\theta, j) - p_j(q) \right\}$$

- Observes state-dependent prices:
  $$p_j(q) = p_j(q_j) = \mathbb{E}[c(w_j)|q_j]$$
- Simplified version of public housing assignment
Stochastic Price Adaptation

Transition if agent arrives, sees queue lengths $q_t$, joins queue $j$
Stochastic Price Adaptation

Transition if item $j$ arrives, assigned to an agent in queue $j$
Stochastic Price Adaptation

- Allocative efficiency $W_{WL}$ is the expected match value under the steady state distribution.
- When there are $>2$ items, the steady state distribution is not tractable.
The Waiting List Mechanism

- The expected allocative efficiency under the waiting list is
  \[ W^{WL} = \mathbb{E}[W(\eta^{WL})] \]

- Adjustment size \( \Delta \) is defined by
  \[ \Delta = \max_{j \in \mathcal{J}} \max_{1 \leq q \leq q_{\text{max}}} \{ p_j(q) - p_j(q - 1) \} \]

- If waiting costs are linear \( c(w) = c \cdot w \), then
  \[ \Delta = \frac{c}{\mu_{\text{min}}} \]
  is the cost of waiting for one item arrival.
Main Result: Bounding Allocative Efficiency

**Theorem 1:**

Allocative efficiency under the waiting list is bounded by

\[ W_{WL} \geq W^{OPT} - \frac{\lambda + 2}{2\lambda} \Delta \]
Main Result: Bounding Allocative Efficiency

**Theorem 1:**
Allocative efficiency under the waiting list is bounded by

\[ W_{WL} \geq W_{OPT} - \frac{\lambda}{2\lambda} \Delta \]

For linear waiting costs, the allocative efficiency loss is bounded by the cost of waiting for one item arrival

- High loss if an apartment arrives monthly, low loss if apartments arrive daily
Main Result: Intuition

Suppose $p^* = \text{cost of waiting six months}$

- If apartments arrives monthly, corresponding queue length is 5
- Each arrival significantly changes the price

- If apartments arrive daily, corresponding queue length is 180
- Each arrival slightly changes the price
Relation to Static Assignment

Let $W^*$ be the optimal allocative efficiency in the corresponding static assignment problem:

$$W^* = \max_{\{x_{\theta j}\}_{\theta \in \Theta, j \in J}} \left\{ \frac{1}{\lambda} \sum_{j \in J} \int_{\Theta} x_{\theta j} v(\theta, j) \, dF(\theta) \right\}$$

subject to

$$\sum_{j \in J} x_{\theta j} \leq 1, \ x_{\theta j} \in [0, 1] \quad \forall \theta \in \Theta$$

$$\int_{\Theta} \lambda x_{\theta j} \, dF(\theta) \leq \mu_j \quad \forall j \in J$$

**Proposition:**

$$W^{OPT} = W^*$$
Duality for the Static Assignment

Lemma (Monge-Kantorovich duality):

\[
\min_{p \geq 0} h(p) = W^*
\]

for

\[
h(p) = \int_\Theta \max_{j \in J \cup \{\emptyset\}} [v(\theta, j) - p_j] + \frac{1}{\lambda} \sum_{j \in J} \mu_j p_j
\]
Relation to Stochastic Gradient Descent

- Let \( p^* \) denote optimal static prices
- Prices \( p(q_t) \) change when an item arrives, or agent arrives
- \( \Delta \) is the maximal adjustment size
Relation to Stochastic Gradient Descent

If prices are \( p(q_t) \) the expected adjustment is

\[
\mathbb{E}[q_{j,t} - q_{j,t+1}] = -\frac{\lambda}{1 + \lambda} \int_{\Theta} 1_{\{a(\theta, q_t) = j\}} dF(\theta) + \frac{1}{1 + \lambda} \mu_j
\]

which is a sub-gradient of the dual objective

\[
h(p) = \int_{\Theta} \max_{j \in J \cup \{\emptyset\}} [v(\theta, j) - p_j] dF(\theta) + \frac{1}{\lambda} \sum_{j \in J} \mu_j p_j
\]

That is, the expected step is in direction of a gradient decent

- Works for deep learning…
- Unlike when SGD is used for optimization, step size \( \Delta \) is fixed and does not shrink to 0
Relation to Stochastic Gradient Descent

- Prices moves towards $p^*$ in expectation
Proof Sketch

- Define a Lyapunov function $L(q)$ such that $\nabla L(q) = p(q)$

- Decompose the value generated from each arrival:

$$
\mathbb{E}[v(\theta_t, a(\theta_t, q_t))|q_t] \geq \frac{\lambda}{\lambda + 1} W^* - L(q_t) - \mathbb{E}[L(q_{t+1})|q_t]
$$

(I) Change in Potential

$$
- \frac{2 + \lambda}{2(1 + \lambda)} \Delta
$$

(II) Loss
Proof Sketch

- Over many periods, the potential term cancels out

\[
\frac{1}{T} \sum_{t=t_0}^{T} [L(q_t) - L(q_{t+1})] = \frac{1}{T} \left( L(q_{t_0}) - L(q_T) \right) \approx 0
\]
Proof Sketch

- Decompose the value generated from each arrival:

\[ \mathbb{E}[v(\theta_t, a(\theta_t, q_t)) | q_t] \geq \frac{\lambda}{\lambda + 1} W^* \]

\[ - L(q_t) - \mathbb{E}[L(q_{t+1}) | q_t] \]

(I) Change in Potential

\[ - \frac{2 + \lambda}{2(1 + \lambda)} \Delta \]

(II) loss

- After canceling (I), the loss per period is bounded by (II)
  - Bound is independent of \( q_t \), implying we do not need to calculate the stationary distribution
When is the Loss High?

**Proposition 2:**
For any number of items $J$ there exist an economy where allocative efficiency is

$$W_{WL} \approx W_{OPT} - \Delta$$
Example of High Loss

- Agents $\Theta = J$, each agent only wants the corresponding item
  \[ \nu(\theta, j) = 1_{\{\theta = j\}} \]

- Identical arrival rates of items and corresponding agents

- Loss when an agent arrives and price is too high (maximal queue length)

- Loss proportional to $\Delta = \frac{c}{\mu_j}$
  - Queue lengths follow an unbiased reflected random walk
  - Queue lengths $q_j = 0, 1, 2, \ldots, 1/\Delta$ equally likely in steady state
  - Probability of hitting the boundary is roughly $\frac{1}{1/\Delta}$.
When is the Loss Low?

Theorem 3:
Consider an economy with finitely many agent types and linear waiting costs \( c(w) = c \cdot w \). Suppose there is a unique market clearing price. Then there exist \( \alpha, \beta, c_0 > 0 \) such that for any \( c < c_0 \)

\[
W_{WL} \geq W_{OPT} - \beta e^{-\alpha/\Delta}
\]

- Note: an economy with finitely many agents generically has a unique market clearing
Theorem 3: Stronger Concentration

- If the dual is unique, no loss within a neighborhood of $p^*$
  - Agents only take items they are assigned under the optimal assignment with positive probability
- Biased random walk towards $p^*$
Optimal Adjustment Size and Price Rigidity

Consider a planner who can set prices, but planner does not know distribution of agent preferences

- Agents arrive over time, can learn from choices of past agents
- Finite horizon $T$

A simple pricing SGD pricing heuristic:

- Increase price of item $j$ by $\Delta$ when an agent chooses $j$
- Decrease the price of item $j$ by $\Delta$ at rate proportional to supply
Optimal Adjustment Size and Price Rigidity

Theorem:
The allocative efficiency of SGD pricing with adjustment size $\Delta = 1/\sqrt{T}$ is at least

$$W_T^{WL} \geq W_T^* - O(\sqrt{T})$$

- Choice of intermediate $\Delta$ balances two sources of loss:
  - Smaller $\Delta$ implies less loss from price fluctuations
  - Larger $\Delta$ implies less transient loss during initial learning

- $O(\sqrt{T})$ is the minimal possible loss (Devanur et al. 2019)
Optimal Adjustment Size and Price Rigidity

Attractive simple pricing heuristic

- Efficiency guarantees
- Algorithm can operate continuously, even if demand changes
- No knowledge required, apart from frequency of changes

Naturally occurring pricing rigidity

- Prices continuously adjust, unaware of changes in demand
  - e.g., do Fed announcements affect demand for Italian food?
- Slow reaction when demand does change
  - Algorithm unsure whether it observes new demand patterns or noise
- No need for menu costs, rational inattention, etc.
Conclusion

- Analysis of allocative efficiency in waiting lists
  - Simple, natural price adaptation process

- Connection to stochastic gradient decent
  - Bounds through Lyapunov functions

- Random fluctuations cause an efficiency loss
  - Simple price adaptation policy can do well
  - Loss depends on the “adjustment size” – how much one arrival changes prices

- Pricing heuristic generates slow response to demand changes